Synthesis of Dissipative Systems Using Quadratic Differential Forms: Part I

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***Abstract*—The problem discussed is that of designing a controller for a linear system that renders a quadratic functional nonnegative. Our formulation and solution of this problem is completely representation-free. The system dynamics are specified by a differential behavior, and the performance is specified through a quadratic differential form. We view control as interconnection: a controller constrains a distinguished set of system variables, the control variables. The resulting behavior of the to-be-controlled variables is called the controlled behavior. The constraint that the controller acts through the control variables only can be succinctly expressed by requiring that the controlled behavior should be wedged in between the hidden behavior, obtained by setting the control variables equal to zero, and the plant behavior, obtained by leaving the control variables unconstrained. The main result is a set of necessary and sufficient conditions for the existence of a controlled behavior that meets the performance specifications. The essential requirement is a coupling condition, an inequality that combines the storage functions of the hidden behavior and the orthogonal complement of the plant behavior.**

***Index Terms*—Behaviors, controller implementability, coupling condition, dissipative systems, hidden behavior, quadratic differential forms, storage functions.**

I. INTRODUCTION

THE subject of this paper is shaping the behavior of a linear system by attaching a controller to it. Conditions are derived that make it possible to render the system dissipative, for example, contractive, or passive. This problem is basically what is usually called the $\mathcal{H}_\infty$-problem. We show that it can be reformulated in an elegant way as that of finding a behavior that is wedged in between two given behaviors and makes a quadratic differential form nonnegative. The "upper bound" results from the fact that the controlled behavior must be physically realizable, and hence included in the (unconstrained) plant behavior. The "lower bound" expresses in a subtle way the restriction that the controlled behavior must be implementable by a controller that acts through the control variables only. The conditions for solvability use the theory of dissipative systems and their associated storage functions. The surprising aspect of the main result is a coupling condition among certain storage functions, more than reminiscent of the clever coupling condition between the solutions of algebraic Riccati equations that first appeared in the classic paper [2]. Our solvability conditions also require the dissipativeness of the hidden behavior and of a suitable orthogonal complement of the plant behavior. These conditions feature prominently also in [4]–[6].

We will cast the development completely in the language of behaviors and the associated quadratic differential forms. This not only allows a clean problem statement, but it results in a formulation that is representation-free and flexible in the algorithms that can be used for verifying the existence and the specification of the controller. Other references where the $\mathcal{H}_\infty$-problem in a behavioral setting has been discussed before are [4], [5], [1], [11], and [10].

A frequently asked question is what the arguments are of approaching problems from a behavioral point of view. The advantages are many. Undoubtedly the most important one is that since the concepts and ideas are representation-free, they allow to treat a wide variety of model classes, more directly emanating from from modeling. In particular, they allow to deal with state space models and transfer functions as special cases of a more general model specification. Behaviors are also much better suited for treating system interconnections. The signal flow graph philosophy that underlies input–output thinking is actually inappropriate for many physical interconnections, for instance for electrical circuits, mechanical systems, fluidic systems, etc. The present paper uses quadratic differential forms in performance criteria, thus also showing what is the appropriate notion for performance specifications of polynomial matrices for system models. In summary, both as a mathematical framework, as well as for dealing with physical systems, the behavioral point of view is simply a richer and more rational setting. As a consequence of this, behavioral concepts are also more easily generalized, witness the recent flurry of activity in this area aimed at PDEs.

A few words about notation. We use the standard notation $\mathbb{R}^{\mathtt{n}}$, $\mathbb{R}^{n_1 \times n_2}$ etc. When a dimension is not specified (but, of course, finite), we write $\mathbb{R}^\bullet$, $\mathbb{R}^{\mathtt{n}\times\bullet}$, $\mathbb{R}^{\bullet\times\bullet}$, etc. We typically use the superscript $\mathtt{w}$ (for example in $\mathbb{R}^{\mathtt{w}}$) when generic elements of that space are denoted by w. The set of real one-variable polynomials in the indeterminate ξ is denoted by $\mathbb{R}[\xi]$ and real rational functions by $\mathbb{R}(\xi)$, with obvious modifications for vectors, matrices, and two-variable polynomials. The set of infinitely differentiable maps from $\mathbb{R}$ to $\mathbb{R}^{\mathtt{n}}$ is denoted by $\mathfrak{C}^\infty(\mathbb{R}, \mathbb{R}^{\mathtt{n}})$, and its subspace consisting of the compact support elements by $\mathfrak{D}(\mathbb{R}, \mathbb{R}^{\mathtt{n}})$. The set of square integrable maps from $\mathbb{R}$ to $\mathbb{R}^{\mathtt{n}}$ is denoted as $\mathcal{L}_2(\mathbb{R}, \mathbb{R}^{\mathtt{n}})$, with obvious modifications for other (co-) domains. Sometimes, when the domain and co-domain are obvious, we simply write $\mathfrak{C}^\infty$, $\mathfrak{D}$, $\mathcal{L}_2$. The $\mathcal{H}_\infty$-norm of a matrix $G \in \mathbb{R}^{\bullet\times\bullet}(\xi)$ is defined as $\|G\|_{\mathcal{H}_\infty} = \sup_{s\in\mathbb{C};\, \mathbb{R}_e(s)\geq 0} \|G(s)\|$. The operator col stacks vectors or matrices; dim denotes dimension, and rowdim, coldim denote, respectively, the number of

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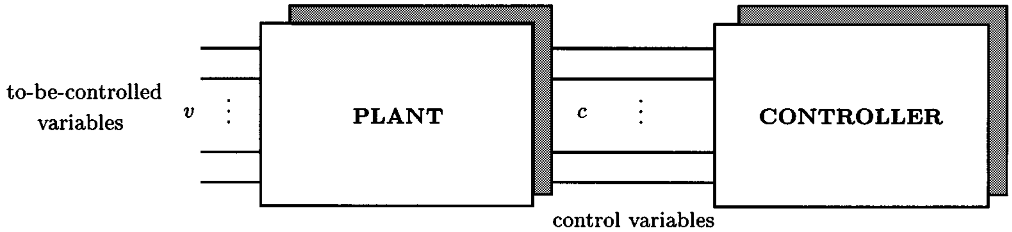

Fig. 1. Plant and controller configuration.

rows or columns of a matrix; $\operatorname{diag}(D_1, D_2, \ldots, D_n)$ forms the block-diagonal matrix with $D_1, D_2, \ldots, D_n$ on the diagonal.

The proofs are collected in Sections VII and VIII.

II. Linear Differential Systems

We have discussed dynamical systems from a behavioral point of view extensively before [7], [13], [14]. We restrict our background remarks in the paper in order to introduce the required concepts and notation and to ensure readability and continuity of the flow of ideas. Also, we introduce the required background material only at the point that it is needed to allow smooth reading.

A subset $\mathfrak{B} \subset \mathfrak{C}^\infty(\mathbb{R}, \mathbb{R}^\bullet$ defines a linear time-invariant differential system (briefly, a *differential system*, or a *differential behavior*) if there exists a polynomial matrix $R \in \mathbb{R}^{\bullet\times\bullet}[\xi]$ such that $\mathfrak{B} = \{w \in \mathfrak{C}^\infty(\mathbb{R}, \mathbb{R}^\bullet) \,|\, R(d/dt)w = 0\}$. By $\mathfrak{L}^\bullet$ we denote the set of linear time-invariant differential systems, and by $\mathfrak{L}^{\mathtt{w}}$ those with $\mathtt{w}$ variables [in other words, with behaviors $\mathfrak{B} \subset \mathfrak{C}^\infty(\mathbb{R}, \mathbb{R}^{\mathtt{w}})$]. This class of systems is a very general one, with nice mathematical structure. It includes finite-dimensional constant linear state systems, systems described by rational transfer functions, or by linear differential equations with auxiliary (latent) variables, etc. Important is to note that while we *define* $\mathfrak{B} \in \mathfrak{L}^\bullet$ as the kernel of a differential operator, $\mathfrak{B}$ is often *not specified* in this way. We speak about a *kernel representation* when $\mathfrak{B} \in \mathfrak{L}^{\mathtt{w}}$ is represented by $R(d/dt)w = 0$, with $\mathfrak{B} = \{w \in \mathfrak{C}^\infty(\mathbb{R}, \mathbb{R}^{\mathtt{w}}) \,|\, R(d/dt)w = 0\}$, the representation through which we have defined $\mathfrak{L}^{\mathtt{w}}$. Another representation is a *latent variable representation*, defined through polynomial matrices R and M by $R(d/dt)w = M(d/dt)\ell$, with $\mathfrak{B} = \{w \in \mathfrak{C}^\infty(\mathbb{R}, \mathbb{R}^{\mathtt{w}} \,|\, \exists\, \ell \in \mathfrak{C}^\infty(\mathbb{R}, \mathbb{R}^\ell)$ such that $R(d/dt)w = M(d/dt)\ell\}$. This is the type of model that usually results from first principles modeling, with the ws the vector of variables that the model aims at, and the ℓs the vector of auxiliary variables introduced in the modeling process (for example state variables). The behavior $\mathfrak{B}$ is then called the *manifest* behavior, and $\mathfrak{B}_{\text{full}} = \{(w, \ell) \in \mathfrak{C}^\infty(\mathbb{R}, \mathbb{R}^{\mathtt{w}+\ell} \,|\, R(d/dt)w = M(d/dt)\ell\}$, the *full behavior*. The fact that $\mathfrak{B} \in \mathfrak{L}^{\mathtt{w}}$ if $\mathfrak{B}_{\text{full}} \in \mathfrak{L}^{\mathtt{w}+1}$, is a direct consequence of the all-important *elimination theorem*. This states the following. Let $\mathfrak{B}' \in \mathfrak{L}^{\mathtt{w}+\mathtt{w}'}$. Define

$$\mathfrak{B} = \{w \in \mathfrak{C}^\infty(\mathbb{R}, \mathbb{R}^{\mathtt{w}}) \,|\, (w, w') \in \mathfrak{B}' \text{ for some } w' \in \mathfrak{C}^\infty(\mathbb{R}, \mathbb{R}^{\mathtt{w}'})\}.$$

Then $\mathfrak{B} \in \mathfrak{L}^{\mathtt{w}}$. The set $\mathfrak{L}^{\mathtt{w}}$ is hence closed under intersection, addition, and projection. Moreover, for $F \in \mathbb{R}^{\mathtt{w}'\times\mathtt{w}}[\xi]$, $\mathfrak{B} \in \mathfrak{L}^{\mathtt{w}}$ implies $F(d/dt)\mathfrak{B} \in \mathfrak{L}^{\mathtt{w}'}$, and $\mathfrak{B}' \in \mathfrak{L}^{\mathtt{w}'}$ implies $(F(d/dt))^{-1}\mathfrak{B}' \in \mathfrak{L}^{\mathtt{w}}$ (this inverse is a set theoretic inverse).

III. Controller Implementability

Consider the linear time-invariant differential plant shown in Fig. 1. It has two types of terminals: terminals carrying *to-be-controlled variables* v and terminals carrying *control variables* c. Assume that there are $\mathtt{v}$ to-be-controlled variables and $\mathtt{c}$ control variables. In the classical controller configuration, the to-be-controlled variables combine the exogenous disturbance inputs and the to-be-controlled outputs, while the control variables combine the sensor outputs and the actuator inputs. A *feedback controller* may be viewed as a signal processor that processes the sensor outputs and returns the actuator inputs. It is the synthesis of such feedback processors that is traditionally viewed as control design. However, we will look at control from a somewhat broader perspective, and we consider any law that restricts the behavior of the control variables as a controller. The motivation of this alternative formulation of control is two-fold. The main motivation is a practical one: many controllers, for example, physical devices as dampers, heat insulators, matched impedances, etc., simply do not act as signal processors. For a more elaborate discussion of this point of view, we refer to [14]. The second motivation is of a theoretical nature. We will see in Theorem 1 that our formulation allows to view control as the design of a behavior that is wedged in between two given behaviors. This is a strikingly simple and appealing formulation indeed.

We now turn to the question what controlled behaviors can be achieved. We refer to this problem as *controller implementation*. The problem may actually be considered as a basic question in engineering design: a behavior is prescribed, and the question is whether this behavior can be achieved by inserting a suitably designed subsystem into the overall system. Before the controller acts, there are two behaviors of the plant that are relevant: the behavior $\mathcal{P}_{\text{full}} \in \mathfrak{L}^{\mathtt{v}+\mathtt{c}}$ (called the *full plant behavior*) of the variables v and c combined, and the behavior $\mathcal{P}$ (called the *plant behavior*) of the to-be-controlled variables v (with the control variables eliminated). Hence

$$\begin{aligned}\mathcal{P}_{\text{full}} &= \{(v, c) \in \mathfrak{C}^\infty(\mathbb{R}, \mathbb{R}^{\mathtt{v}+\mathtt{c}}) \,|\, (v, c) \\ &\qquad \text{satisfies the plant equations}\} \\ \mathcal{P} &= \{v \in \mathfrak{C}^\infty(\mathbb{R}, \mathbb{R}^{\mathtt{v}}) \,|\, \exists\, c \in \mathfrak{C}^\infty(\mathbb{R}, \mathbb{R}^{\mathtt{c}}) \\ &\qquad \text{such that } (v, c) \in \mathcal{P}_{\text{full}}\}.\end{aligned}$$

By the elimination theorem, $\mathcal{P} \in \mathfrak{L}^{\mathtt{v}}$. The controller restricts the control variables c and (assuming that it is a linear time-invariant differential system) is described by a *controller behavior* $\mathcal{C} \in \mathfrak{L}^{\mathtt{c}}$. Hence

$$\mathcal{C} = \{c \in \mathfrak{C}^\infty(\mathbb{R}, \mathbb{R}^{\mathtt{c}}) \,|\, c \text{ satisfies the controller equations}\}.$$

After the controller is attached, we obtain the *controlled behavior* $\mathcal{K}$ defined by

$$\mathcal{K} = \{v \in \mathfrak{C}^\infty(\mathbb{R}, \mathbb{R}^{\mathtt{v}} \mid \exists\, c \in \mathcal{C} \text{ such that } (v, c) \in \mathcal{P}_{\text{full}}\}.$$

Note that, again by the elimination theorem, $\mathcal{K} \in \mathfrak{L}^{\mathtt{v}}$. We say that $\mathcal{C}$ *implements* $\mathcal{K}$ if the above relation holds between $\mathcal{C}$ and $\mathcal{K}$.

We now discuss the following question:

$$\boxed{\begin{array}{l}\text{For what } \mathcal{K} \in \mathfrak{L}^{\mathtt{v}} \text{ does there exists a}\\ \mathcal{C} \in \mathfrak{L}^{\mathtt{c}} \text{ that implements } \mathcal{K}?\end{array}}.$$

This question has a very simple and elegant answer: it depends only on the manifest plant behavior $\mathcal{P}$ and on the behavior consisting of the plant trajectories with the control variables put equal to zero. This behavior is denoted by $\mathcal{N}$, and is called the *hidden behavior*. It is defined as

$$\mathcal{N} = \{v \in \mathcal{P} \mid (v, 0) \in \mathcal{P}_{\text{full}}\}.$$

Theorem 1 (Controller Implementability Theorem): Let $\mathcal{P}_{\text{full}} \in \mathfrak{L}^{\mathtt{v}+\mathtt{c}}$ be the full plant behavior, $\mathcal{P} \in \mathfrak{L}^{\mathtt{v}}$ the manifest plant behavior, and $\mathcal{N} \in \mathfrak{L}^{\mathtt{v}}$ the hidden behavior. Then $\mathcal{K} \in \mathfrak{L}^{\mathtt{v}}$ is implementable by a controller $\mathcal{C} \in \mathfrak{L}^{\mathtt{c}}$ acting on the control variables if and only if

$$\boxed{\mathcal{N} \subset \mathcal{K} \subset \mathcal{P}.}$$

Theorem 1 shows that $\mathcal{K}$ can be *any* behavior that is wedged in between the given behaviors $\mathcal{N}$ and $\mathcal{P}$. The necessity of this condition is quite intuitive: $\mathcal{K} \subset \mathcal{P}$ states that the controlled behavior must be part of the plant behavior. Logical, since the controller merely restricts what can happen. The condition $\mathcal{K} \supset \mathcal{N}$ states that the behavior $\mathcal{N}$ must remain possible, whatever be the controller. This is quite intuitive also, since the subbehavior of the plant behavior that is compatible with $c = 0$, hence when the controller receives no information on what is happening in the plant, must remain possible in the controlled behavior, whatever controller is chosen. This observation has important consequences in control: in order for there to exist a controller that achieves acceptable performance, the hidden behavior must already meet the specifications, since *there is simply no way to eliminate it by means of control*. The fact that the hidden behavior must meet the control specifications has been observed before in a $\mathcal{H}_\infty$ context for example in [3], [4], and [6]. A noteworthy special case is $\mathcal{N} = 0$. This means that the to-be-controlled variables are observable (in the precise way this term is used in the behavioral context) from the control variables. We refer to this condition as *full information control*. In this case, any sub-behavior $\mathcal{K} \in \mathfrak{L}^{\mathtt{v}}$ of $\mathcal{P}$ is implementable.

Theorem 1 reduces control problems to finding the controlled behavior $\mathcal{K}$ directly. Of course, the problem of how to actually implement $\mathcal{K}$ needs to be addressed at some point. In particular, the question when a particular controlled behavior can be implemented by a feedback processor remains a very important one, and will be discussed in Part II.

IV. Main Problem Formulation

In this section, we state in a self-contained form, the problem treated in this paper and the main result. The control specification is expressed by a quadratic functional in the to-be-controlled variables whose integral needs to be nonnegative, expressing, for instance, disturbance attenuation and stability of the controlled system.

A. Preamble

In order to formulate this problem mathematically, we need a few more preliminaries about differential systems, quadratic differential forms, and dissipative systems. We have taken a few shortcuts with regard to the definition of behaviors, dissipativity, storage functions, etc. In particular, the solutions of the differential equations under consideration are assumed to be infinitely differentiable. We also assume that the systems that we deal with (the plant, the controlled behavior, the hidden behavior, etc.) are controllable. Next, when dealing with infinite integrals, we often assume, to avoid convergence issues, that the trajectories have compact support. For this reason, it is convenient to introduce the notation $\mathfrak{B} \cap \mathfrak{D} := \mathfrak{B} \cap \mathfrak{D}(\mathbb{R}, \mathbb{R})^\bullet$. These assumptions are made partly for convenience of exposition. We will briefly mention later how our results should be adapted when we consider solutions in $\mathcal{L}_2^{\text{loc}}$, when the controllability conditions are not met, and when we include trajectories in $\mathcal{L}_2$ in the definition of a dissipative system.

We call $\mathfrak{B} \in \mathfrak{L}^\bullet$ *controllable* if for all $w_1, w_2 \in \mathfrak{B}$, there exists a $T \geq 0$ and a $w \in \mathfrak{B}$ such that $w(t) = w_1(t)$ for $t < 0$ and $w(t+T) = w_2(t)$ for $t \geq 0$. Denote the controllable elements of $\mathfrak{L}^\bullet$, $\mathfrak{L}^{\mathtt{w}}$ by $\mathfrak{L}^\bullet_{\text{cont}}$, $\mathfrak{L}^{\mathtt{w}}_{\text{cont}}$. For controllable systems (only), $\mathfrak{B} \cap \mathfrak{D}$ specifies $\mathfrak{B} \in \mathfrak{L}^\bullet$ uniquely: $\mathfrak{B}$ is the $\mathfrak{C}^\infty$-closure of $\mathfrak{B} \cap \mathfrak{D}$ if and only if $\mathfrak{B}$ is controllable.

Behaviors $\mathfrak{B} \in \mathfrak{L}^\bullet$ are described by a differential equation of the form $R(d/dt)w = 0$, typically with $\operatorname{rowdim}(R) < \operatorname{coldim}(R)$. Mathematically, $R(d/dt)w = 0$ is then called an *under-determined* system of equations. This results in the fact that some of the components of $w = (w_1, w_2, \ldots, w_{\mathtt{w}})$ are unconstrained. The number of unconstrained components, an integer "invariant" associated with $\mathfrak{B}$, is called the *input cardinality*. It is defined by the map $\mathtt{m}\colon \mathfrak{L}^\bullet \to \mathbb{Z}_+$ that associates with $\mathfrak{B} \in \mathfrak{L}^\bullet$, $\mathtt{m}(\mathfrak{B})$, its number of free, "input," variables ("input" can be interpreted intuitively in the usual sense see [7] and Section VI-A). It is easily proven that the system $\mathfrak{B} \in \mathfrak{L}^{\mathtt{w}}$ described by $R(d/dt)w = 0$ has input cardinality $\mathtt{m}(\mathfrak{B}) = \mathtt{w} - \operatorname{rank}(R)$.

We use the abbreviations BF for "bilinear form," QF for "quadratic form," BDF for "bilinear differential form," and QDF for "quadratic differential form." QFs play an important role in linear system theory: as performance criteria, as Lyapunov functions, etc. In the context of behavioral differential systems, quadratic functionals are most naturally formulated as BDFs and QDFs. These notions are key elements in the behavioral approach to control. They are now briefly introduced (more details can be found in Section VI-C and in [15]). In the present section, we only consider the elements that are needed in the formulation of the main problem that we discuss in this paper.

The QF on $\mathbb{R}^\bullet$ induced by the matrix $S = S^T \in \mathbb{R}^{\bullet\times\bullet}$ is denoted by $q_S(x) = |x|_S^2 = x^T S x$. When $S = I$, the subscript in $|x|_S^2$ is usually deleted. Denote the signature of S by $\text{sign}(S) = (\sigma_-(S), \sigma_+(S))$, with $\sigma_-(S)$ and $\sigma_+(S)$ the number of negative and positive eigenvalues of S respectively. Of course, $\text{sign}(S) = \text{sign}(q_S)$ (see Section VI-C for the notion of the signature of a QF). The $\mathcal{L}_2$-norm of $x \in \mathcal{L}_2(\mathbb{R}, \mathbb{R}^\bullet)$, $\sqrt{(\int_{-\infty}^{+\infty} |x|^2\, dt)}$, is denoted by $\|x\|_{\mathcal{L}_2(\mathbb{R}, \mathbb{R}^\bullet)}$. Note that single bars refer to norms on Euclidean spaces, while double bars refer to norms on $\mathcal{L}_2$. BF's and QF's in the setting of differential systems are parametrized very effectively by two-variable polynomial matrices. Let $\Phi \in \mathbb{R}^{\mathtt{w}_1 \times \mathtt{w}_2}[\zeta, \eta]$, written out in terms of its coefficient matrices as the (finite) sum $\Phi(\zeta, \eta) = \Sigma_{k, \ell \in \mathbb{Z}_+} \Phi_{k, \ell} \zeta^k \eta^\ell$. It induces the map L_Φ: $\mathfrak{C}^\infty(\mathbb{R}, \mathbb{R}^{\mathtt{w}_1}) \times \mathfrak{C}^\infty(\mathbb{R}, \mathbb{R}^{\mathtt{w}_2}) \to \mathfrak{C}^\infty(\mathbb{R}, \mathbb{R})$, defined by

$$L_\Phi(w_1, w_2) = \sum_{k, \ell \in \mathbb{Z}_+} \left(\frac{d^k}{dt^k} w_1\right)^T \Phi_{k, \ell} \left(\frac{d^\ell}{dt^\ell} w_2\right).$$

This map is called the *bilinear differential form (BDF)* induced by Φ. When $\mathtt{w}_1 = \mathtt{w}_2 = \mathtt{w}$, L_Φ induces the map Q_Φ: $\mathfrak{C}^\infty(\mathbb{R}, \mathbb{R}^{\mathtt{w}}) \mapsto \mathfrak{C}^\infty(\mathbb{R}, \mathbb{R})$, defined by $Q_\Phi(w) = L_\Phi(w, w)$, i.e.,

$$Q_\Phi(w) = \sum_{k, \ell \in \mathbb{Z}_+} \left(\frac{d^k}{dt^k} w\right)^T \Phi_{k, \ell} \left(\frac{d^\ell}{dt^\ell} w\right).$$

This map is called the *quadratic differential form (QDF)* induced by Φ. Denote $\Phi^T(\eta, \zeta)$ as $\Phi^*(\zeta, \eta)$. Note that when considering QDFs, we may as well assume that Φ defining Q_Φ is symmetric, that is $\Phi = \Phi^*$, i.e., $\Phi_{k\ell} = \Phi_{\ell k}^T$ for all $k, \ell \in \mathbb{Z}_+$. Indeed, $Q_\Phi = Q_{\Phi^*} = Q_{(1/2)(\Phi + \Phi^*)}$ entails symmetry without loss of generality.

Let $\Phi = \Phi^* \in \mathbb{R}^{\mathtt{w} \times \mathtt{w}}[\zeta, \eta]$ and $\mathfrak{B} \in \mathfrak{L}^{\mathtt{w}}_{\text{cont}}$. The system $\mathfrak{B}$ is said to be *dissipative with respect to* Q_Φ, (briefly, Φ-dissipative) if $\int_{-\infty}^{+\infty} Q_\Phi(w)\, dt \geq 0$ for all $w \in \mathfrak{B} \cap \mathfrak{D}$. It is said to be *dissipative on* $\mathbb{R}_-$ with respect to Q_Φ, (briefly, Φ-dissipative on $\mathbb{R}_-$) if $\int_{-\infty}^{0} Q_\Phi(w)\, dt \geq 0$ for all $w \in \mathfrak{B} \cap \mathfrak{D}$. Dissipativity on $\mathbb{R}_+$ is analogously defined. Obviously, dissipativity on $\mathbb{R}_-$ or $\mathbb{R}_+$ implies dissipativity. As we shall see in Section IV-C, dissipativity on $\mathbb{R}_-$ combines dissipativity on the whole of $\mathbb{R}$ with stability.

For an intuitive interpretation, identify $Q_\Phi(w)(t)$ with the power, the rate of energy, delivered to the system at time t, and $\int_{-\infty}^{+\infty} Q_\Phi(w)\, dt$ with the total net energy delivered to the system by taking it through the history w. Dissipativity states that the system absorbs energy during any history in $\mathfrak{B}$ that starts and ends with the system at rest. Dissipativity on $\mathbb{R}_-$ states that *at any time* the net flow of energy *up to that time* has been *into* the system.

Note that the definition of dissipativeness makes perfect sense for QDFs that involve derivatives in the variables. However, in this paper we only consider dissipative systems with respect to Q_Φ with constant $\Phi = \Phi^T \in \mathbb{R}^{\mathtt{w} \times \mathtt{w}}$. Note that in this case $Q_\Phi(w) = |w|_\Phi^2$. We have limited our definition of dissipativeness to controllable systems. Obtaining a satisfactory generalization of this notion to noncontrollable systems (yielding for example a nice synthesis for passive electrical circuits) is still a matter of ongoing research.

B. Problem Formulation

Equipped with the notions of the behavior of a differential system, of the input cardinality, of the signature of a symmetric matrix, and of a dissipative system, we are able to formulate the mathematical problem that we will solve in this paper.

The QDF Q_Σ with $\Sigma = \Sigma^T \in \mathbb{R}^{\mathtt{v} \times \mathtt{v}}$ defines the *weighting functional* that enters in the control performance. Denote by $\text{sign}(\Sigma) = (\sigma_-(\Sigma), \sigma_+(\Sigma))$ its signature. The problem that we solve in this paper may succinctly be formulated as follows:

PROBLEM FORMULATION:

Let $\mathcal{N}, \mathcal{P} \in \mathfrak{L}^{\mathtt{v}}_{\text{cont}}$, and $\Sigma = \Sigma^T \in \mathbb{R}^{\mathtt{v} \times \mathtt{v}}$ nonsingular; $\mathcal{P}$ is called the *plant behavior*, $\mathcal{N}$ the *hidden behavior*, and Q_Σ the *weighting functional*. The problem is to find $\mathcal{K} \in \mathfrak{L}^{\mathtt{v}}_{\text{cont}}$ (called the *controlled behavior*) such that:

1. $\mathcal{N} \subset \mathcal{K} \subset \mathcal{P}$ $\quad$ (*implementability*),
2. $\mathcal{K}$ is Σ-dissipative on $\mathbb{R}_-$ $\quad$ (*dissipativity*),
3. $\mathtt{m}(\mathcal{K}) = \sigma_+(\Sigma)$ $\quad$ (*liveness*).

We now explain informally the interpretation of these conditions. The first condition has been explained in Theorem 1. The inclusion $\mathcal{K} \subset \mathcal{P}$ signifies that the controlled behavior is physically possible: the controller merely restricts the plant behavior. We view this as *realizability*. The inclusion $\mathcal{K} \supset \mathcal{N}$ is more subtle. It means that the controlled behavior is implementable by a controller that acts through the control variables only.

That the controlled behavior $\mathcal{K}$ must be Σ-dissipative is the basic control design specification. As is well-known, by suitably choosing Σ, it implies disturbance attenuation, or passivation. The fact that Σ-dissipativity is required to hold on $\mathbb{R}_-$, and not just on $\mathbb{R}$, implies stability of the controlled behavior (see Section IV-C).

The liveness requirement states that $\sigma_+(\Sigma)$ components of v must remain free in the controlled behavior. It expresses that the controlled system must still be able to accept free exogenous inputs: the controlled behavior is not allowed to restrict the exogenous inputs directly, it only serves to shape the influence of the exogenous inputs on the endogenous outputs. The following proposition shows that the liveness condition is equivalent to the requirement that in the controlled behavior there are as many free variables as possible.

Proposition 2: Let $\mathfrak{B} \in \mathfrak{L}^{\mathtt{w}}_{\text{cont}}$ and $\Phi = \Phi^T \in \mathbb{R}^{\mathtt{w} \times \mathtt{w}}$ be nonsingular. Assume that $\mathfrak{B}$ is Φ-*dissipative*. Then $\mathtt{m}(\mathfrak{B}) \leq \sigma_+(\Phi)$.

The problem statement can thus be rephrased as

When does there exist there a controlled behavior that is Σ-dissipative on $\mathbb{R}_-$ and of maximal input cardinality?

C. Examples

We now illustrate the problem formulation by means of important examples.

1) **Disturbance attenuation.** In the important case of $\mathcal{H}_\infty$-disturbance attenuation we have $v = (d, f)$ with d exogenous disturbance inputs, f endogenous to-be-controlled outputs, and $Q_\Sigma(v) = |d|^2 - |f|^2$, whence $\Sigma = \mathrm{diag}(I_{\mathtt{d}}, -I_{\mathtt{f}})$. In this case $\mathcal{K} \in \mathfrak{L}^{\mathtt{v}}_{\mathrm{cont}}$ is Σ-dissipative on $\mathbb{R}_-$ and $\mathtt{m}(\mathcal{K}) = \sigma_+(\Sigma)(= \mathtt{d})$ if and only if in $\mathcal{K}$, d is input, f is output and the $\mathcal{H}_\infty$-norm of the transfer function from d to f in $\mathcal{K}$, $G_{d\mapsto f}$, satisfies $\|G_{d\mapsto f}\|_{\mathcal{H}_\infty} \leq 1$ (see Section VI-A for a definition of the notion of transfer function in a behavioral setting, and Part II for a proof of these claims).
2) **Passivation.** A similar story holds, with disturbance attenuation replaced by passivity, when $v = (e, f)$ (e for "effort," f for "flow") and $Q_\Sigma(e, f) = e^T f$, with $e_k f_k$ the "power" flowing into the plant through the kth exogenous port or terminal, whence $\Sigma = (1/2)\left[\begin{smallmatrix} 0 & I_{\mathtt{e}} \\ I_{\mathtt{e}} & 0 \end{smallmatrix}\right]$. In this case $\mathcal{K} \in \mathfrak{L}^{\mathtt{v}}_{\mathrm{cont}}$ is Σ-dissipative on $\mathbb{R}_-$, and $\mathtt{m}(\mathcal{K}) = \sigma_+(\Sigma)(= \mathtt{e} = \mathtt{f})$ if and only if there is a component-wise input–output partition (u, y) of $v = (e, f)$ such that for all $1 \leq i \leq \mathtt{e} = \mathtt{f}$, either e_i, or f_i, is input, and the other is output, and the transfer function from u to y in $\mathcal{K}$, $G_{u\mapsto y}$, is positive real, i.e., $G_{u\mapsto y}(\lambda) + G^T_{u\mapsto y}(\bar{\lambda}) \geq 0$ for all $\lambda \in \mathbb{C}$, with $\mathrm{Re}(\lambda) > 0$ (see Part II for a proof of these claims).

   Of course, in these examples, dissipativity on $\mathbb{R}_-$ leads to stability robustness for terminations along the v-terminals that satisfy the small gain or the passive operator conditions. These well-known implications to stability robustness of controlled systems is one of the main motivations of the problem discussed in this paper.
3) **Frequency weighting.** Another performance specification that fits in our problem formulation is to consider $|d|^2 - |f|^2$, with d and f related to the "physical" exogenous input disturbance d' and endogenous to-be-controlled output f' by $Q(d/dt)d' = P(d/dt)d$, $N(d/dt)f = D(d/dt)f'$, with P and D square, nonsingular, and Hurwitz. The dynamical relations between d and d', and between f and f' allow frequency weighting, while the Hurwitz assumptions allow to conclude from the stability $d \in \mathcal{L}_2 \Rightarrow f \in \mathcal{L}_2$ (that will result from dissipativity on $\mathbb{R}_-$), the desired stability $d' \in \mathcal{L}_2 \Rightarrow f' \in \mathcal{L}_2$ of the controlled system. In passivation, such QDF's with derivatives allow to consider expressions for the power as $F^T(d/dt)q$. These occur in mechanical systems, with F the force, and q the position, and (F, q) the to-be-controlled variables. In the present paper, we have limited our attention to QDF's in the performance of the controlled system of the form Q_Σ with Σ a constant matrix. Of course, it is of interest to be able to treat performances of the form Q_Σ with $\Sigma = \Sigma^* \in \mathbb{R}^{\mathtt{v}\times\mathtt{v}}[\zeta, \eta]$ directly, without rewriting this QDF in terms of a constant matrix (which is always possible, see Section VI-C). We will deal with this in a sequel paper.

V. Main Results

In order to state the solution to the problem formulated above, we need a couple of more preliminaries: the notion of a storage function, and orthogonality of behaviors.

A. Storage Functions

Let $\mathfrak{B} \in \mathfrak{L}^{\mathtt{w}}$, $\Phi \in \Phi^* \in \mathbb{R}^{\mathtt{w}\times\mathtt{w}}[\zeta, \eta]$, and $\Psi = \Psi^* \in \mathbb{R}^{\mathtt{w}\times\mathtt{w}}[\zeta, \eta]$. Then Q_Ψ is said to be a *storage function for* $\mathfrak{B}$ *with respect to the supply rate* Q_Φ if the *dissipation inequality* $(d/dt)Q_\Psi(w) \leq Q_\Phi(w)$ holds for all $w \in \mathfrak{B}$. For $f\colon A \to \mathbb{R}$, $f \geq 0$ means $f(t) \geq 0\, \forall\, t \in A$. There is an immediate relation between dissipativity and the existence of a storage function, with its sign related to half-line dissipativity.

Proposition 3: Let $\mathfrak{B} \in \mathfrak{L}^{\mathtt{w}}_{\mathrm{cont}}$ and $\Phi = \Phi^* \in \mathbb{R}^{\mathtt{w}\times\mathtt{w}}[\zeta, \eta]$. Then $\mathfrak{B}$ is Φ-dissipative if and only if there exists $\Psi = \Psi^* \in \mathbb{R}^{\mathtt{w}\times\mathtt{w}}[\zeta, \eta]$ such that Q_Ψ is a storage function for $\mathfrak{B}$ with respect to the supply rate Q_Φ. Furthermore, $\mathfrak{B}$ is Φ-dissipative on $\mathbb{R}_-$ if and only if Q_Ψ can be taken to be nonnegative on $\mathfrak{B}$, i.e., $Q_\Psi(w) \geq 0$ for all $w \in \mathfrak{B}$, and Φ-dissipative on $\mathbb{R}_+$ if and only if Q_Ψ can be taken to be nonpositive on $\mathfrak{B}$.

The theme of the above proposition is a recurrent one: it identifies a "global" statement (dissipativity: an inequality involving an integral over $\mathbb{R}$) with a "local" statement (the dissipation inequality: an inequality that is *point-wise* on $\mathbb{R}$). Intuitively, the proposition states that a system globally dissipates supply along any trajectory on the whole of $\mathbb{R}$ if and only if this dissipation can be brought into evidence through a storage function whose rate of increase does not exceed, point-wise in time, the rate of supply delivered to the system. The storage function is far from unique, but much is known about the set of possible storage functions. We will return to this in Section V-D.

B. Orthogonal Behaviors

We also need the orthogonal complement of a controllable behavior, with orthogonality viewed with respect to a BDF induced by a constant matrix. Let $\Phi \in \mathbb{R}^{\mathtt{w}\times\mathtt{w}}$, and $\mathfrak{B}_1, \mathfrak{B}_2 \in \mathfrak{L}^{\mathtt{w}}_{\mathrm{cont}}$; $\mathfrak{B}_1$ and $\mathfrak{B}_2$ are said to be *orthogonal with respect to* L_Φ (briefly, Φ-*orthogonal*) if $\int_{-\infty}^{+\infty} L_\Phi(w_1, w_2)\, dt = 0$ for all $w_1 \in \mathfrak{B}_1 \cap \mathfrak{D}$ and $w_2 \in \mathfrak{B}_2 \cap \mathfrak{D}$. We denote this as $\mathfrak{B}_1 \perp_\Phi \mathfrak{B}_2$. Let $\mathfrak{B} \in \mathfrak{L}^{\mathtt{w}}_{\mathrm{cont}}$, and define the Φ-orthogonal complement $\mathfrak{B}^{\perp_\Phi}$ of $\mathfrak{B}$ as

$$\mathfrak{B}^{\perp_\Phi} = \left\{ w \in \mathfrak{C}^\infty(\mathbb{R}, \mathbb{R}^{\mathtt{w}}) \,\middle|\, \int_{-\infty}^{+\infty} L_\Phi(w, w')\, dt = 0 \right.$$
$$\left. \text{for all } w' \in \mathfrak{B} \cap \mathfrak{D} \right\}.$$

It is easy to see that $\mathfrak{B}^{\perp_\Phi} \in \mathfrak{L}^{\mathtt{w}}_{\mathrm{cont}}$. When $\Phi = I_{\mathtt{w}}$, we denote $\perp_\Phi$ simply as $\perp$. Note that $\mathfrak{B}^{\perp_\Phi} = (\Phi\mathfrak{B})^\perp = (\Phi^T)^{-1}\mathfrak{B}^\perp$ (-1 denotes the set-theoretic inverse), and, if Φ is nonsingular, then $\mathfrak{B} = (\mathfrak{B}^{\perp_\Phi})^{\perp_{(\Phi^T)^{-1}}}$.

In order to state our main result, we need the following proposition.

Proposition 4: Let $\Phi \in \mathbb{R}^{\mathtt{w}\times\mathtt{w}}$, and $\mathfrak{B}_1, \mathfrak{B}_2 \in \mathfrak{L}^{\mathtt{w}}_{\mathrm{cont}}$. Then there exists a $\Psi \in \mathbb{R}^{\mathtt{w}\times\mathtt{w}}[\zeta, \eta]$ such that $(d/dt)L_\Psi(w_1, w_2) = L_\Phi(w_1, w_2)$ for all $w_1 \in \mathfrak{B}_1$ and $w_2 \in \mathfrak{B}_2$, if and only if $\mathfrak{B}_1 \perp_\Phi \mathfrak{B}_2$. Moreover, Ψ is essentially unique, in the sense that if $\Psi_1, \Psi_2 \in \mathbb{R}^{\mathtt{w}\times\mathtt{w}}[\zeta, \eta]$ both satisfy this equality, then $L_{\Psi_1}(w_1, w_2) = L_{\Psi_2}(w_1, w_2)$ for all $w_1 \in \mathfrak{B}_1$ and $w_2 \in \mathfrak{B}_2$.

The idea of the above proposition is again the equivalence of a "local" and a "global" property, this time for orthogonality. We call the BDF L_Ψ (or simply Ψ) of this proposition, $[(\mathfrak{B}_1, \mathfrak{B}_2);\ \Phi]$-*adapted*.

C. Main Result

Equipped with these additional preliminaries, the notion of a storage function and existence of an adapted BDF, we are able to state the solution of the problem formulated in Section IV-B. This problem allows an explicit and representation-free solution, involving the storage functions associated with dissipative systems in a subtle way.

Theorem 5 (Main Results): The controlled behavior $\mathcal{K} \in \mathfrak{L}^{\mathtt{v}}_{\text{cont}}$ described in the problem formulation exists if and only if the following conditions are satisfied:

1. $\mathcal{N}$ is Σ-dissipative,
2. $\mathcal{P}^{\perp_\Sigma}$ is $(-\Sigma)$-dissipative,
3. there exist $\Psi_{\mathcal{N}}, \Psi_{\mathcal{P}^{\perp_\Sigma}}, \Psi_{(\mathcal{N}, \mathcal{P}^{\perp_\Sigma})} \in \mathbb{R}^{\mathtt{v}\times\mathtt{v}}[\zeta, \eta]$, defining
   - a storage function $Q_{\Psi_{\mathcal{N}}}$ for $\mathcal{N}$ as a Σ-dissipative system, i.e., $\frac{d}{dt} Q_{\Psi_{\mathcal{N}}}(v_1) \le Q_\Sigma(v_1)$ for $v_1 \in \mathcal{N}$,
   - a storage function $Q_{\Psi_{\mathcal{P}^{\perp_\Sigma}}}$ for $\mathcal{P}^{\perp_\Sigma}$ as a $(-\Sigma)$-dissipative system, i.e., $\frac{d}{dt} Q_{\Psi_{\mathcal{P}^{\perp_\Sigma}}}(v_2) \le -Q_\Sigma(v_2)$ for $v_2 \in \mathcal{P}^{\perp_\Sigma}$,
   - and the $[(\mathcal{N}, \mathcal{P}^{\perp\Sigma});\ \Sigma]$-adapted BDF $L_{\Psi_{(\mathcal{N}, \mathcal{P}^{\perp_\Sigma})}}$, i.e., $\frac{d}{dt} L_{\Psi_{(\mathcal{N}, \mathcal{P}^{\perp_\Sigma})}}(v_1, v_2) = L_\Sigma(v_1, v_2)$, for $v_1 \in \mathcal{N}, v_2 \in \mathcal{P}^{\perp_\Sigma}$,

   such that that the QDF
$$\boxed{Q_{\Psi_{\mathcal{N}}}(v_1) - Q_{\Psi_{\mathcal{P}^{\perp_\Sigma}}}(v_2) + 2L_{\Psi_{(\mathcal{N}, \mathcal{P}^{\perp_\Sigma})}}(v_1, v_2)} \tag{1}$$
   is nonnegative for all $v_1 \in \mathcal{N}$ and $v_2 \in \mathcal{P}^{\perp_\Sigma}$.

Note that the storage functions in (1) are well-defined by the assumed dissipativeness of $\mathcal{N}$ and $\mathcal{P}^{\perp_\Sigma}$, and that $\Psi_{(\mathcal{N}, \mathcal{P}^{\perp_\Sigma})}$ is well-defined by Proposition 4, since $\mathcal{N} \subset \mathcal{P}$.

The surprising condition in the above result is the required nonnegativity of (1). We refer to this condition as the *coupling condition*. It implies in particular that $Q_{\Psi_{\mathcal{N}}}$ is nonnegative on $\mathcal{N}$, which shows that $\mathcal{N}$ is Σ-dissipative on $\mathbb{R}_-$, clearly (since $\mathcal{N} \subset \mathcal{K}$) a necessary condition for the existence of $\mathcal{K}$. It also implies that $Q_{\Psi_{\mathcal{P}^{\perp_\Sigma}}}$ is nonpositive on $\mathcal{P}^{\perp_\Sigma}$, which in turn shows that $\mathcal{P}^{\perp_\Sigma}$ is $(-\Sigma)$-dissipative on $\mathbb{R}_+$. It is not difficult to see that this is also a necessary condition for the existence of $\mathcal{K}$. In fact, it can be shown (see Proposition 12) that Σ-dissipativity of $\mathcal{K}$ on $\mathbb{R}_-$ combined with $\mathtt{m}(\mathcal{K}) = \sigma_+(\Sigma)$, implies that $\mathcal{K}^{\perp_\Sigma}$ is $(-\Sigma)$-dissipative on $\mathbb{R}_+$. In addition, $\mathcal{K} \subset \mathcal{P}$ implies $\mathcal{P}^{\perp_\Sigma} \subset \mathcal{K}^{\perp_\Sigma}$. Therefore $(-\Sigma)$-dissipativity of $\mathcal{P}^{\perp_\Sigma}$ on $\mathbb{R}_+$ is also a necessary condition for the existence of $\mathcal{K}$. As already mentioned, both these elements of the solution are present in [4] and [5]. What makes (1) a surprising result is the coupling of $Q_{\Psi_{\mathcal{N}}}$ and $Q_{\Psi_{\mathcal{P}^{\perp_\Sigma}}}$ through $L_{\Psi_{(\mathcal{N}, \mathcal{P}^{\perp_\Sigma})}}$, thus strengthening the required nonnegativity of the storage functions $Q_{\Psi_{\mathcal{N}}}$ and $-Q_{\Psi_{\mathcal{P}^{\perp_\Sigma}}}$, and coupling the dissipativeness of $\mathcal{N}$ and $\mathcal{P}^{\perp_\Sigma}$. This condition is analogous to (but a representation-free generalization of) the remarkable condition coupling solutions of algebraic Riccati equations that first appeared in the instant-classic paper [2].

Our main result, Theorem 5, is stated merely as an existence result. Since storage functions are in an essential way nonunique, the theorem leaves unanswered which storage functions yield likely candidates for satisfaction of the coupling condition (1). Next, we state a result that avoids this drawback, but, in order to do so, we need some details on the set of storage functions.

D. The Available Storage and the Required Supply

As stated in Proposition 3, $\mathfrak{B} \in \mathfrak{L}^{\mathtt{w}}_{\text{cont}}$ is Φ-dissipative if and only if there exists a storage function Q_Ψ. There are many storage functions. For example, if Ψ_1 and Ψ_2 both induce storage functions, then so does their convex combination $\alpha\Psi_1 + (1-\alpha)\Psi_2$ for $0 \le \alpha \le 1$. Important for our aims, however, is the existence of *extreme* storage functions, as stated in the following proposition.

Proposition 6: Let $\mathfrak{B} \in \mathfrak{L}^{\mathtt{w}}_{\text{cont}}$ and $\Phi = \Phi^* \in \mathbb{R}^{\mathtt{w}\times\mathtt{w}}[\zeta, \eta]$, and assume that $\mathfrak{B}$ is Φ-dissipative. Then there exist storage functions $Q_{\Psi^{\text{inf}}}$ and $Q_{\Psi^{\text{sup}}}$ induced by $\Psi^{\text{inf}}, \Psi^{\text{sup}} \in \mathbb{R}^{\mathtt{w}\times\mathtt{w}}[\zeta, \eta]$, such that for any other storage function Q_Ψ, there holds $Q_{\Psi^{\text{inf}}}(w) \le Q_\Psi(w) \le Q_{\Psi^{\text{sup}}}(w)$ for all $w \in \mathfrak{B}$.

These extreme storage functions are respectively called the *available storage* ($Q_{\Psi^{\text{inf}}}$) and the *required supply* ($Q_{\Psi^{\text{sup}}}$). This nomenclature stems from the following variational interpretation as storage functions generated by trajectories that maximize the supply extracted from, respectively, minimize the supply delivered to, a system

$$Q_{\Psi^{\text{inf}}}(w)(0) = \sup\left(-\int_0^\infty Q_\Phi(\tilde{w})\,dt\right)$$

where the supremum is taken over all $\tilde{w} \in \mathfrak{B}$ such that $w \wedge \tilde{w} \in \mathfrak{B}$ ($\wedge$ denotes *concatenation*: $w_1 \wedge w_2$ is defined by $(w_1 \wedge w_2)(t) := w_1(t)$ for $t < 0$, and $(w_1 \wedge w_2)(t) = w_2(t)$ for $t \ge 0$), and

$$Q_{\Psi^{\text{sup}}}(w)(0) = \inf\left(\int_{-\infty}^0 Q_\Phi(\tilde{w})\,dt\right)$$

where the infimum is taken over all $\tilde{w} \in \mathfrak{B}$ such that $\tilde{w} \wedge w \in \mathfrak{B}$. The supremum and infimum in these expressions should be understood as follows. We are considering a particular $w \in \mathcal{B}$ and would like to find out what $Q_{\Psi^{\text{inf}}}(w)$ and $Q_{\Psi^{\text{sup}}}(w)$ are. Obviously, using shift-invariance, it suffices to specify $Q_{\Psi^{\text{inf}}}(w)(0)$ and $Q_{\Psi^{\text{sup}}}(w)(0)$. These formulas show how to interpret Ψ^{inf} and Ψ^{sup}: fix the past (or future) of a trajectory to being that of w, concatenate with the future (or past) of any other trajectory

$\tilde{w} \in \mathfrak{B}$, and take the supremum (or infimum) over all $\tilde{w}$ such that $w \wedge \tilde{w}$ (or $\tilde{w} \wedge w$) also belongs to $\mathfrak{B}$.

E. An Alternative Formulation of the Main Result

We now reformulate Theorem 5 using extreme storage functions

Theorem 7 (Alternative formulation of the main main): The controlled behavior $\mathcal{K} \in \mathfrak{L}^{\mathrm{v}}_{\mathrm{cont}}$ described in the problem formulation exists if and only if the following conditions are satisfied:

1. $\mathcal{N}$ is Σ-dissipative,
2. $\mathcal{P}^{\perp_\Sigma}$ is $(-\Sigma)$-dissipative,
3. the QDF

$$\boxed{Q_{\Psi^{\mathrm{sup}}_{\mathcal{N}}}(v_1) - Q_{\Psi^{\mathrm{inf}}_{\mathcal{P}^{\perp_\Sigma}}}(v_2) + 2L_{\Psi_{(\mathcal{N},\, \mathcal{P}^{\perp_\Sigma})}}(v_1,\, v_2)} \tag{2}$$

is nonnegative for all $v_1 \in \mathcal{N}$ and $v_2 \in \mathcal{P}^{\perp_\Sigma}$, where

- $\Psi^{\mathrm{sup}}_{\mathcal{N}} \in \mathbb{R}^{\mathrm{v}\times\mathrm{v}}[\zeta,\, \eta]$ induces the required supply $Q_{\Psi^{\mathrm{sup}}_{\mathcal{N}}}$ for $\mathcal{N}$ as a Σ-dissipative system,
- $\Psi^{\mathrm{inf}}_{\mathcal{P}^{\perp_\Sigma}} \in \mathbb{R}^{\mathrm{v}\times\mathrm{v}}[\zeta,\, \eta]$ induces the available storage $Q_{\Psi^{\mathrm{inf}}_{\mathcal{P}^{\perp_\Sigma}}}$ for $\mathcal{P}^{\perp_\Sigma}$ as a $(-\Sigma)$-dissipative system,
- $\Psi_{(\mathcal{N},\, \mathcal{P}^{\perp_\Sigma})} \in \mathbb{R}^{\mathrm{v}\times\mathrm{v}}[\zeta,\, \eta]$ induces the $[(\mathcal{N},\, \mathcal{P}^{\perp_\Sigma});\ \Sigma]$-adapted BDF $L_{\Psi_{(\mathcal{N},\, \mathcal{P}^{\perp_\Sigma})}}$.

The QDF (2) is uniquely defined on $\mathcal{N} \times \mathcal{P}^{\perp_\Sigma}$.

The main result in its above form is a fully explicit condition for the existence of the desired controlled behavior $\mathcal{K}$, since $\Psi^{\mathrm{sup}}_{\mathcal{N}}$, $\Psi^{\mathrm{inf}}_{\mathcal{P}^{\perp_\Sigma}}$, and $\Psi_{(\mathcal{N},\, \mathcal{P}^{\perp_\Sigma})}$ can readily be computed from representations of $\mathcal{P}$ and $\mathcal{N}$. This will be illustrated for the case that $\mathcal{P}_{\mathrm{full}}$ is given in state-space representation in Part II. Additional computational aspects will be discussed elsewhere, and are based on LMIs, AREs, and spectral factorization, and their generalization to behavioral representations and QDFs. The algorithms form an interplay between one- and two variable polynomial matrices. Because of length limitations, we are unable to deal with algorithms here. They will be the subject of a follow-up paper.

We end this section with a result about the dynamic order of the controlled behavior. We denote by $\mathtt{n}(\mathfrak{B})$ the dimension of the minimal state representation of $\mathfrak{B}$ (see Section VI-A).

Theorem 8: Let $\mathcal{N}, \mathcal{P} \in \mathfrak{L}^{\mathrm{v}}_{\mathrm{cont}}$, and let $\Sigma = \Sigma^T \in \mathbb{R}^{\mathrm{v}\times\mathrm{v}}$ be nonsingular. If there exists a controlled behavior $\mathcal{K} \in \mathfrak{L}^{\mathrm{v}}_{\mathrm{cont}}$ such that $\mathcal{N} \subset \mathcal{K} \subset \mathcal{P}$, $\mathcal{K}$ is Σ-dissipative on $\mathbb{R}_-$, and $\mathtt{m}(\mathcal{K}) = \sigma_+(\Sigma)$, then there exists such a $\mathcal{K}$ with $\mathtt{n}(\mathcal{K}) \leq \mathtt{n}(\mathcal{N}) + \mathtt{n}(\mathcal{P})$.

It is easy to prove that $\mathtt{n}(\mathcal{N}), \mathtt{n}(\mathcal{P}) \leq \mathtt{n}(\mathcal{P}_{\mathrm{full}})$. The bound given above therefore yields, in particular, $\mathtt{n}(\mathcal{K}) \leq 2\mathtt{n}(\mathcal{P}_{\mathrm{full}})$. In the *full information case* discussed in Part II, we obtain, in fact, $\mathtt{n}(\mathcal{K}) \leq \mathtt{n}(\mathcal{P}_{\mathrm{full}})$.

VI. Background Material

In this section, we collect the additional background material that is used in the proofs.

A. More on Differential Systems

We have already discussed kernel and latent variable representations. In the proofs we also use image and state representations. These are now introduced, but first we review the notion of observability. Assume that the signal space is a product space, with the first component w_1 *observed* variables, and the second component, w_2 *to-be-deduced* variables. Then w_2 is said to be *observable from* w_1 *in* $\mathfrak{B} \in \mathfrak{L}^{\mathrm{w}_1+\mathrm{w}_2}$ if $(w_1,\, w_2')$, $(w_1,\, w_2'') \in \mathfrak{B}$ implies $w_2' = w_2''$. Observability is equivalent to the existence of $F \in \mathbb{R}^{\mathrm{w}_2\times\mathrm{w}_1}[\xi]$ such that $(w_1,\, w_2) \in \mathfrak{B}$ implies $w_2 = F(d/dt)w_1$. We call a latent variable system $R(d/dt)w = M(d/dt)\ell$ *observable* if the latent variables ℓ are observable from the manifest variables w in the full behavior $\mathfrak{B}_{\mathrm{full}}$. This is the case if and only if there exists a $D \in \mathbb{R}^{\dim(\ell)\times\mathrm{w}}(\xi)$ such that $(w,\, \ell) \in \mathfrak{B}_{\mathrm{full}}$ implies $\ell = D(d/dt)w$. Equivalently, if and only if M has a polynomial left inverse, M^L, hence $M^L(\xi)M(\xi) = I$, in which case $\ell = M^L(d/dt)R(d/dt)w$ recovers ℓ from $w \in \mathfrak{B}$.

A very useful characterization of controllable systems is that they are precisely the systems that admit an *image representation*, a latent variable representation of the form $w = M(d/dt)\ell$, with manifest behavior $\mathfrak{B} = \mathrm{im}(M(d/dt))$. It can be shown that $\mathfrak{B} \in \mathfrak{L}^{\bullet}_{\mathrm{cont}}$ if and only if it admits an image representation, in fact, if and only if it admits an observable image representation. The *controllable part* of a behavior is defined as follows. Let $\mathfrak{B} \in \mathfrak{L}^{\mathrm{w}}$. There exists $\mathfrak{B}' \in \mathfrak{L}^{\mathrm{w}}_{\mathrm{cont}}$, $\mathfrak{B}' \subset \mathfrak{B}$ such that $\mathfrak{B}'' \in \mathfrak{L}^{\mathrm{w}}_{\mathrm{cont}}$, $\mathfrak{B}'' \subset \mathfrak{B}$ implies $\mathfrak{B}'' \subset \mathfrak{B}'$, i.e, $\mathfrak{B}'$ is the largest controllable sub-behavior contained in $\mathfrak{B}$. Denote this system as $\mathfrak{B}_{\mathrm{cont}}$. It can be shown that $\mathfrak{B}_{\mathrm{cont}}$ is the closure in the $\mathfrak{C}^\infty$-topology of $\mathfrak{B} \cap \mathfrak{D}$.

Let $\mathfrak{B} \in \mathfrak{L}^{\mathrm{w}}$. Then there exists a permutation of the components of the vector $w = (w_1,\, w_2,\, \ldots,\, w_{\mathrm{w}(\mathfrak{B})})$ of system variables, such that it can be divided into the two sub-vectors $w = (u,\, y)$, with u free, and y bound. This means that for any $u \in \mathfrak{C}^\infty(\mathbb{R},\, \mathbb{R}^{\dim(u)})$, there exists a finite-dimensional affine subspace of $y \in \mathfrak{C}^\infty(\mathbb{R},\, \mathbb{R}^{\dim(y)})$, such that $(u,\, y) \in \mathfrak{B}$. Equivalently, $u \in \mathfrak{C}^\infty(\mathbb{R},\, \mathbb{R}^{\dim(u)})$ and $(d^k/dt^k)y(0)$ for $k \in \mathbb{Z}_+$ then specify the $y \in \mathfrak{C}^\infty(\mathbb{R},\, \mathbb{R}^{\dim(y)})$ such that $(u,\, y) \in \mathfrak{B}$ uniquely. Moreover, $\mathtt{m}(\mathfrak{B}) = \dim(u)$, but the input–output partition itself is not unique. A kernel representation is an *input–output representation* $P(d/dt)y = Q(d/dt)u$, $w = (u,\, y)$ if P is square and $\det(P) \neq 0$. We refer to the matrix of rational functions $P^{-1}Q$ as the *transfer function* from u to y, and denote it as $G_{u\mapsto y}$.

The notion of state occurs naturally in the context of behavioral systems, and has been extensively discussed in [8] and [12]. Even though it did not enter the problem formulation nor its solution, it is used forcefully in the proofs. A state system is a latent variable system, in which the latent variable has the *property of state*, i.e., if $(w_1,\, x_1)$, $(w_2,\, x_2) \in \mathfrak{B}_{\mathrm{full}}$ are such that $x_1(0) = x_2(0)$, then $(w_1,\, x_1) \wedge (w_2,\, x_2)$, the concatenation of $(w_1,\, x_1)$ and $(w_2,\, x_2)$ at $t = 0$, belongs to the closure (in the topology of $\mathcal{L}^{\mathrm{loc}}$) of $\mathfrak{B}_{\mathrm{full}}$. A state system is said to be (state) *trim* if for all $x_0 \in \mathbb{R}^{\dim(x)}$, there exists $(w, x) \in \mathfrak{B}_{\text{full}}$ such that $x(0) = x_0$. It is said to be *minimal* if the state has minimal dimension among all state representations that have the same manifest behavior. It can be shown that a state system is minimal if and only if it is trim and observable. In particular, there then exists $X \in \mathbb{R}^{\dim(x)\times \mathtt{w}}(\xi)$ such that $x = X(dt/dt)w$ for $(w, x) \in \mathfrak{B}_{\text{full}}$ and such that for all $x_0 \in \mathbb{R}^{\dim(x)}$, there exists $w \in \mathfrak{B}$ such that $x_0 = X(d/dt)w(0)$. The polynomial matrix $X \in \mathbb{R}^{\bullet\times \mathtt{w}}[\xi]$ is said to induce a *state map* for $\mathfrak{B} \in \mathfrak{L}^{\mathtt{w}}$ if $X(d/dt)w$ is a state for $\mathfrak{B}$, meaning that the full behavior $\{(w, X(d/dt)w) \mid w \in \mathfrak{B}\}$ has the state property. A state map is minimal if and only if it induces a trim state representation.

A latent variable system has the state property if and only if its full behavior can be represented by a differential equation that is zeroth order in w and first order in x, i.e., by $R_0 w = M_0 x + M_1(d/dt)x$, with R_0, M_0, M_1 constant matrices. There are many, more structured, state representations as, for instance, a *driving variable representation* $(d/dt)x = Ax + Bd$, $w = Cx + Dd$, with d an, obviously free, additional latent variable; an *output nulling representation* $(d/dt)x = Ax + Bw$, $0 = Cx + Dw$; or an *input-state-output representations* $(d/dt)x = Ax + Bu$, $y = Cx + Du$, $w = (u, y)$, the most popular of them all. Every system $\mathfrak{B} \in \mathfrak{L}^{\bullet}$ admits such a representation after a suitable permutation of the components of w and a suitable choice of the state.

There are a number of important *integer "invariants"* associated with behaviors. In Section II, we already discussed the input cardinality $\mathtt{m}$, i.e., the number of input components, or free variables. Other integer invariants are the number of manifest variables itself, the number of output components, and the number of state variables. These are formally given by the maps $\mathtt{w}, \mathtt{m}, \mathtt{p}, \mathtt{n}\colon \mathfrak{L}^{\bullet} \mapsto \mathbb{N}$ defined by

$$
\begin{aligned}
\mathtt{w}(\mathfrak{B}) &= \mathtt{w}, \qquad \text{if } \mathfrak{B} \in \mathfrak{L}^{\mathtt{w}}\\
\mathtt{m}(\mathfrak{B}) &= \text{the number of free (input) variables in} \mathfrak{B}\\
\mathtt{p}(\mathfrak{B}) &= \mathtt{w}(\mathfrak{B}) - \mathtt{m}(\mathfrak{B}), \text{ the number of bound (output)}\\
&\quad \text{variables in } \mathfrak{B}\\
\mathtt{n}(\mathfrak{B}) &= \text{the minimal number of state variables}\\
&\quad \text{(the } McMillan\ degree\text{) of } \mathfrak{B}.
\end{aligned}
$$

A state map $X(d/dt)$ is thus minimal if and only if $\operatorname{rowdim}(X) = \mathtt{n}(\mathfrak{B})$. The input cardinality $\mathtt{m}(\mathfrak{B})$ equals the number of inputs in every input–output, input/state/output, or driving variable representation with a minimal number of driving variables. For $\mathfrak{B} \in \mathfrak{L}^{\bullet}_{\text{cont}}$, it also equals the number of latent variables in any observable image representation of $\mathfrak{B}$. The output cardinality equals $\operatorname{rank}(R)$ of any kernel representation of $\mathfrak{B}$, and the number of outputs in every input–output or input/state/output representation. Let $\mathfrak{B}_{\text{cont}}$ be the controllable part of $\mathfrak{B}$. Then $\mathtt{m}(\mathfrak{B}_{\text{cont}}) = \mathtt{m}(\mathfrak{B})$, $\mathtt{n}(\mathfrak{B}_{\text{cont}}) \le \mathtt{n}(\mathfrak{B})$, with $\mathtt{n}(\mathfrak{B}_{\text{cont}}) = \mathtt{n}(\mathfrak{B})$ if and only if $\mathfrak{B}$ is controllable. Also, $\mathfrak{B}' \in \mathfrak{L}^{\mathtt{w}}_{\text{cont}}$, $\mathfrak{B}' \subset \mathfrak{B}$, and $\mathtt{m}(\mathfrak{B}') = \mathtt{m}(\mathfrak{B})$ imply $\mathfrak{B}' = \mathfrak{B}_{\text{cont}}$. The following result is used in the proofs.

Proposition 9: For $\mathfrak{B}_1, \mathfrak{B}_2 \in \mathfrak{L}^{\mathtt{w}}$, $\mathtt{m}(\mathfrak{B}_1 + \mathfrak{B}_2) = \mathtt{m}(\mathfrak{B}_1) + \mathtt{m}(\mathfrak{B}_2) - \mathtt{m}(\mathfrak{B}_1 \cap \mathfrak{B}_2)$.

These representations and integer invariants of a behavior and of its orthogonal complement are closely related. In particular, $\mathbb{R}(d/dt)w = 0$ is a kernel representation of $\mathfrak{B} \in \mathfrak{L}^{\bullet}_{\text{cont}}$ if and only if $w = R^T(-d/dt)\ell$ is an image representation of $\mathfrak{B}^{\perp}$. Also, $R(d/dt)w = 0$ is a full row rank kernel representation of $\mathfrak{B} \in \mathfrak{L}^{\bullet}_{\text{cont}}$ if and only if $w = R^T(-d/dt)\ell$ is an observable image representation of $\mathfrak{B}^{\perp}$. Their driving variable and output nulling representations are related as follows: $(d/dt)x = Ax + Bu$, $w = Cx + Du$ is a driving variable representation of $\mathfrak{B}$ if and only if $(d/dt)z = -A^T z + C^T w$, $0 = B^T z - D^T w$ is an output nulling representation of $\mathfrak{B}^{\perp}$. Their input/state/output representations are related as follows: $(d/dt)x = Ax + Bw_1$, $w_2 = Cx + Dw_1$, $w = (w_1, w_2)$ is an input/state/output representation of $\mathfrak{B}$ if and only if $(d/dt)z = -A^T z + C^T w_1'$, $w_2' = B^T z - D^T w_1'$, $w' = \operatorname{col}(w_1', w_2')$ is an input/state/output representation of $\mathfrak{B}^{\perp}$. It follows that $\mathtt{m}(\mathfrak{B}) = \mathtt{p}(\mathfrak{B}^{\perp})$, $\mathtt{p}(\mathfrak{B}) = \mathtt{m}(\mathfrak{B}^{\perp})$, and $\mathtt{n}(\mathfrak{B}) = \mathtt{n}(\mathfrak{B}^{\perp})$. It also shows that for any $X \in \mathbb{R}^{\mathtt{n}(\mathfrak{B})\times \mathtt{w}}[\xi]$ that induces a minimal state map for $\mathfrak{B}$, there exists $Z \in \mathbb{R}^{\mathtt{n}(\mathfrak{B})\times \mathtt{w}}[\xi]$ that induces a minimal state map for $\mathfrak{B}^{\perp}$, such that

$$\frac{d}{dt}\left(X\left(\frac{d}{dt}\right)w_1\right)^T \left(Z\left(\frac{d}{dt}\right)w_2\right) = w_1^T w_2$$

for all $w_1 \in \mathfrak{B}$ and $w_2 \in \mathfrak{B}^{\perp}$. This yields in particular Proposition 4. We call a pair of polynomial matrices (X, Z) that induce minimal state maps for $\mathfrak{B}$, $\mathfrak{B}^{\perp}$ and satisfy the above equality, a *matched pair* of minimal state maps for $(\mathfrak{B}, \mathfrak{B}^{\perp})$. Hence, if (X, Z) is a matched pair of minimal state maps for $(\mathfrak{B}, \mathfrak{B}^{\perp})$ and Φ a constant square matrix, then L_Ψ with $\Psi(\zeta, \eta) = X^T(\zeta)Z(\eta)\Phi$ yields a $[(\mathfrak{B}, \mathfrak{B}^{\perp_\Phi});\ \Phi]$-adapted BDF.

A system $\mathfrak{B} \in \mathfrak{L}^{\bullet}$ is said to be *memoryless* if $w_1, w_2 \in \mathfrak{B}$ implies that the concatenation $w_1 \wedge w_2$ belongs to the $\mathcal{L}^{\text{loc}}$-closure of $\mathfrak{B}$. Obviously $\mathfrak{B}$ is memoryless if and only if it admits a kernel representation $R_0 w = 0$ with $R_0 \in \mathbb{R}^{\bullet\times\bullet}$. Let $\mathfrak{B} \in \mathfrak{L}^{\bullet}$ and assume that $X(d/dt)$ induces a minimal state map for it. The *memoryless part* of $\mathfrak{B} \in \mathfrak{L}^{\mathtt{w}}$, denoted $\mathfrak{B}_{\text{memoryless}}$, is defined as $\mathfrak{B}_{\text{memoryless}} = \{w \in \mathfrak{B} \mid X(d/dt)w = 0\}$. It is easy to see that $\mathfrak{B}_{\text{memoryless}} \in \mathfrak{L}^{\mathtt{w}}$ and that it is memoryless. In terms of an output nulling representation of $\mathfrak{B}$, it is described by $0 = Bw$, $0 = Dw$. This is a kernel representation of $\mathfrak{B}_{\text{memoryless}}$ and shows that $\mathtt{m}(\mathfrak{B}_{\text{memoryless}}) = \dim(\ker(\operatorname{col}(B, D)))$.

B. More on Dissipative Systems

The notion of state discussed in Section VI-A can be interpreted intuitively as *information state*. It formalizes the memory of a system. For dissipative systems, we have met another notion that is intuitively also related to the memory: the storage function. These two concepts, state and storage, are intimately connected, as explained in the following proposition.

Proposition 10: Let $\mathfrak{B} \in \mathfrak{L}^{\mathtt{w}}_{\text{cont}}$ and let $X \in \mathbb{R}^{\bullet\times \mathtt{q}}[\xi]$ induce a state map for $\mathfrak{B}$. Assume that $\mathfrak{B}$ is Φ-dissipative with $\Phi = \Phi^T \in \mathbb{R}^{\mathtt{w}\times\mathtt{w}}$. Let $\Psi \in \mathbb{R}^{\mathtt{w}\times\mathtt{w}}[\zeta, \eta]$ induce a storage function Q_Ψ for $\mathfrak{B}$ as a Φ-dissipative system. Then Q_Ψ is a memoryless function of the state, i.e., there exists a matrix $K = K^T \in \mathbb{R}^{\operatorname{rowdim}(X)\times\operatorname{rowdim}(X)}$ such that $Q_\Psi(w) = |X(d/dt)w|_K^2$ for $w \in \mathfrak{B}$. In particular, $\mathfrak{B}$ is Φ-dissipative on $\mathbb{R}_-$ if and only if there exists $K = K^T \in \mathbb{R}^{\mathtt{n}(\mathfrak{B})\times\mathtt{n}(\mathfrak{B})}$, $K \ge 0$, such that $|X(d/dt)w|_K^2$ is a storage function.

The following corollary is an immediate consequence of this proposition.

Corollary 11: Let $\mathfrak{B}_1, \mathfrak{B}_2 \in \mathfrak{L}^{\mathtt{w}}_{\text{cont}}$ and $\Phi = \Phi^T \in \mathbb{R}^{\mathtt{w}\times\mathtt{w}}$, and assume that $\mathfrak{B}_1 \subset \mathfrak{B}_2^{\perp_\Phi}$. Assume that $\Psi \in \mathbb{R}^{\mathtt{w}\times\mathtt{w}}[\zeta, \eta]$ is such that the BDF $\mathfrak{L}_\Psi$ induces a $[(\mathfrak{B}_1, \mathfrak{B}_2);\ \Phi]$-adapted BDF, i.e., $(d/dt)\mathfrak{L}_\Psi(w_1, w_2) = w_1^T\Phi w_2$ for $w_1 \in \mathfrak{B}_1$ and $w_2 \in \mathfrak{B}_2$ (by proposition 4 such a Ψ exists). Assume that $X_1, X_2 \in \mathbb{R}^{\bullet\times\mathtt{w}}[\xi]$ induce state maps for $\mathfrak{B}_1$ and $\mathfrak{B}_2$, respectively. Then there exists a matrix $N \in \mathbb{R}^{\text{rowdim}(X_1)\times\text{rowdim}(X_2)}$ such that

$$L_\Psi(w_1, w_2) = \left(X_1\left(\frac{d}{dt}\right)w_1\right)^T N\left(X_2\left(\frac{d}{dt}\right)w_2\right)$$

for $w_1 \in \mathfrak{B}_1$ and $w_2 \in \mathfrak{B}_2$.

In general, there is no immediate relation between dissipativity of $\mathfrak{B}$ and $\mathfrak{B}^{\perp_\Phi}$, unless $\mathtt{m}(\mathfrak{B}) = \sigma_+(Q_\Phi)$. The following proposition deals with this very relevant special case.

Proposition 12: Let $\mathfrak{B} \in \mathfrak{L}^{\mathtt{w}}_{\text{cont}}$ and $\Phi = \Phi^T \in \mathbb{R}^{\mathtt{w}\times\mathtt{w}}$ be nonsingular. Assume that $\mathtt{m}(\mathfrak{B}) = \sigma_+(\Phi)$. Then

1) $\mathfrak{B}$ is Φ-dissipative if and only if $\mathfrak{B}^{\perp_\Phi}$ is $(-\Phi)$-dissipative.
2) $\mathfrak{B}$ is Φ-dissipative on $\mathbb{R}_-$ if and only if $\mathfrak{B}^{\perp_\Phi}$ is $(-\Phi)$-dissipative on $\mathbb{R}_+$.
3) If $\mathfrak{B}$ is Φ-dissipative on $\mathbb{R}_-$, then every storage function Q_Ψ for $\mathfrak{B}$ as a Φ-dissipative system satisfies $Q_\Psi(w) \geq 0$ for $w \in \mathfrak{B}$, and every storage function $Q_{\Psi'}$ for $\mathfrak{B}^{\perp_\Phi}$ as a $(-\Phi)$-dissipative system satisfies $Q_{\Psi'}(w) \leq 0$ for $w \in \mathfrak{B}^{\perp_\Phi}$.
4) Let (X, Z) be a matched pair of minimal state maps for $(\mathfrak{B}, \mathfrak{B}^\perp)$. If $|X(d/dt)w|_K^2$ is a storage function for $\mathfrak{B}$ as a Φ-dissipative system, with $K = K^T \in \mathbb{R}^{\mathtt{n}(\mathfrak{B})\times\mathtt{n}(\mathfrak{B})}$ nonsingular, then $-|Z(d/dt)\Phi w|_{K^{-1}}^2$ is a storage function for $\mathfrak{B}^{\perp_\Phi}$ as a $(-\Phi)$-dissipative system.
5) Moreover, if $\mathfrak{B}$ is Φ-dissipative on $\mathbb{R}_-$, then every $K = K^T \in \mathbb{R}^{\mathtt{n}(\mathfrak{B})\times\mathtt{n}(\mathfrak{B})}$ such that $|X(d/dt)w|_K^2$ is a storage function satisfies $K > 0$.

We call the pair of storage functions $(Q_\Psi, Q_{\Psi'})$, for respectively $\mathfrak{B}$ and $\mathfrak{B}^{\perp_\Phi}$, with $Q_\Psi(w) = |X(d/dt)w|_K^2$, $Q_{\Psi'}(w) = |Z(d/dt)\Phi w|_P^2$, (X, Z) a matched pair of minimal state maps for $(\mathfrak{B}, \mathfrak{B}^\perp)$, and with $P = -K^{-1}$, a *Φ-matched pair of storage functions*. This matching of storage functions is guaranteed when $\mathtt{m}(\mathfrak{B}) = \sigma_+(\Phi)$ and K is nonsingular, the latter certainly being the case if $\mathfrak{B}$ is Φ-dissipative on $\mathbb{R}_-$.

C. More on Quadratic Forms

QFs and QDFs play a central part in the proofs. We therefore introduce them in some detail, and from a rather abstract point of view. A *bilinear form (BF)* on the real vector spaces $(\mathbb{V}_1, \mathbb{V}_2)$ is a mapping ℓ: $\mathbb{V}_1 \times \mathbb{V}_2 \to \mathbb{R}$ that is linear in both arguments. When $\mathbb{V}_1 = \mathbb{V}_2 = \mathbb{V}$, we call it a BF on $\mathbb{V}$. The *dual* of ℓ, ℓ^*, is the BF on $\mathbb{V}_2 \times \mathbb{V}_1$ defined by $\ell^*(v_1, v_2) := \ell(v_2, v_1)$. The BF ℓ on $\mathbb{V}$ is said to be *symmetric* if $\ell = \ell^*$. The BF ℓ on $\mathbb{V}$ induces through $q(x) := \ell(x, x)$ the *quadratic form (QF)* q: $\mathbb{V} \to \mathbb{R}$ on $\mathbb{V}$. Clearly, ℓ, ℓ^*, and $(1/2)(\ell + \ell^*)$ induce the same QF. The *rank* of a BF equals the number of independent linear functionals $\ell(\cdot, v_2)$ where v_2 ranges over $\mathbb{V}_2$, equivalently the number of independent linear functionals $\ell(v_1, \cdot)$, where v_1 ranges over $\mathbb{V}_1$. The rank of a QF equals the rank of the symmetric BF that induces it.

The expression $\sum_{k=1}^{n_+}|f_k^+(x)|^2 - \sum_{k=1}^{n_-}|f_k^-(x)|^2$, with the f_k^+s and f_k^-s linear functionals on $\mathbb{V}$ defines a QF on $\mathbb{V}$. In fact, a QF can be expressed in this way if and only if its rank is finite. If $f_1^+, f_2^+, \ldots, f_{n_+}^+, f_1^-, f_2^-, \ldots, f_{n_-}^-$ are independent (in which case both n_- and n_+ are individually minimal over all such decompositions of $q|_{\mathbb{V}}$ as a sum and difference of squares), then the pair of nonnegative integers (n_-, n_+) is called the *signature* of $q|_{\mathbb{V}}$, and denoted as $\text{sign}(q|_{\mathbb{V}}) = (\sigma_-(q|_{\mathbb{V}}), \sigma_+(q|_{\mathbb{V}}))$. The rank of $q|_{\mathbb{V}}$ equals $\sigma_-(q|_{\mathbb{V}}) + \sigma_+(q|_{\mathbb{V}})$. A QF on $\mathbb{R}^{\mathtt{n}}$ is always of the form $x \mapsto x^TSx$ for some $S = S^T \in \mathbb{R}^{\mathtt{n}\times\mathtt{n}}$. We call this the QF induced by S. The rank and the signature of this QF are equal to that of S, defined as the pair $(\sigma_-(S), \sigma_+(S))$ consisting of the number of negative and positive eigenvalues of S. If $S = S^T \in \mathbb{R}^{\mathtt{n}\times\mathtt{n}}$ and $\mathbb{V}$ is a linear subspace of $\mathbb{R}^{\mathtt{n}}$, then $S|\mathbb{V}$ denotes the QF on $\mathbb{V}$ defined by $x \mapsto x^TSx$. If V is a matrix such that $\mathbb{V} = \text{im}(V)$, then $\text{sign}(S|_{\mathbb{V}}) = \text{sign}(V^TSV)$. The following matrix lemma relates the signatures of the QFs induced by a matrix Q on the linear subspaces $\mathbb{L}$ and $\mathbb{L}^{\perp_Q} = \{x \in \mathbb{R}^{\mathtt{n}} \,|\, x^TQ\mathbb{L} = 0\}$. These relations play an important role in the proof of the main theorem.

Lemma 13: Let $\mathbb{L} \subset \mathbb{R}^{\mathtt{n}}$ be a linear subspace and $Q = Q^T \in \mathbb{R}^{\mathtt{n}\times\mathtt{n}}$. There holds

$$\begin{aligned}\sigma_-\left(Q|_{\mathbb{L}^{\perp_Q}}\right) &= \sigma_+(Q|_{\mathbb{L}}) + \sigma_-(Q) - \dim(\mathbb{L})\\ &\quad + \dim(\mathbb{L}\cap\ker(Q))\\ \sigma_+\left(Q|_{\mathbb{L}^{\perp_Q}}\right) &= \sigma_-(Q|_{\mathbb{L}}) + \sigma_+(Q) - \dim(\mathbb{L})\\ &\quad + \dim(\mathbb{L}\cap\ker(Q)).\end{aligned}$$

We now discuss BFs and QFs in the context of differential behaviors. There is a one-to-one relation between the BDF L_Φ and the BF on $\mathfrak{C}^\infty(\mathbb{R}, \mathbb{R}^{\mathtt{w}_1}) \times \mathfrak{C}^\infty(\mathbb{R}, \mathbb{R}^{\mathtt{w}_2})$ defined by $(w_1, w_2) \mapsto L_\Phi(w_1, w_2)(0)$, and between the QDF Q_Φ and the QF on $\mathfrak{C}^\infty(\mathbb{R}, \mathbb{R}^{\mathtt{w}})$, defined by $w \mapsto Q_\Phi(w)(0)$. The ranks and signatures of a BDF or QDF are defined by this correspondence. Both L_Φ and Q_Φ are of finite rank, although they act on infinite-dimensional spaces. This can be seen as follows. Associate with $\Phi(\zeta, \eta) = \Sigma_{k,\ell\in\mathbb{Z}_+}\Phi_{k,\ell}\zeta^k\eta^\ell \in \mathbb{R}^{\mathtt{w}_1\times\mathtt{w}_2}[\zeta, \eta]$, the matrix $\text{mat}(\Phi)$, defined as the infinite block-matrix whose $(k+1, \ell+1)$th block equals $\Phi_{k,\ell}$, and with the one-variable polynomial matrix $P(\xi) = \Sigma_{k\in\mathbb{Z}_+}P_k\xi^k$, the block-column matrix $\text{mat}(P)$, defined as the infinite block matrix whose $(k+1)$th block equals P_k. These matrices, while infinite, have only a finite number of nonzero entries, and behave like finite matrices. It is easy to see that the rank and the signature of the QF defined by Q_Φ with $\Phi = \Phi^*$ is equal to that of the symmetric matrix $\text{mat}(\Phi)$ [defined as the rank and the signature of a truncation of $\text{mat}(\Phi)$ that deletes only zeros]. Clearly $\text{mat}(\Phi)$ can be factored as $\text{mat}(\Phi) = \Gamma_+^T\Gamma_+ - \Gamma_-^T\Gamma_-$, with Γ_+ and Γ_- infinite matrices with a finite number of rows, such that $(\text{rowdim}(\Gamma_-), \text{rowdim}(\Gamma_+)) = \text{sign}(Q_\Phi)$, equivalently, with the rows of $\text{col}(\Gamma_+, \Gamma_-)$ linearly independent over $\mathbb{R}$. Define the polynomial matrices $F_+, F_- \in \mathbb{R}^{\bullet\times\mathtt{w}}[\xi]$ through $\Gamma_+ = \text{mat}(F_+)$, i.e., $F_+(\xi) = \Gamma_+\text{col}(I_{\mathtt{w}}, I_{\mathtt{w}}\xi, I_{\mathtt{w}}\xi^2, \ldots)$, and $\Gamma_- = \text{mat}(F_-)$. Then $\Phi(\zeta, \eta) = F_+^T(\zeta)F_+(\eta) - F_-^T(\zeta)F_-(\eta)$. Hence $\Phi(\zeta, \eta)$ can be factored in terms of one-variable polynomial matrices as $\Phi(\zeta, \eta) = F_+^T(\zeta)F_+(\eta) - F_-^T(\zeta)F_-(\eta)$ with the rows of $F =$

$\mathrm{col}(F_+, F_-) \in \mathbb{R}^{\bullet\times\bullet}[\xi]$ linearly independent over $\mathbb{R}$. Such a factorization of Φ is called a *canonical factorization*. A canonical factorization yields the signature and the rank of Q_Φ by $\mathrm{sign}(Q_\Phi) = (\sigma_-(Q_\Phi), \sigma_+(Q_\Phi)) = (\mathrm{rowim}(F_-), \mathrm{rowdim}(F_+))$, and $\mathrm{rank}(Q_\Phi) = \mathrm{rowdim}(F)$. Hence $\mathrm{sign}(Q_\Phi)$ equals the number of negative and positive squares in the factorization $Q_\Phi(w) = |F_+(d/dt)w|^2 - |F_-(d/dt)w|^2$ as a sum and difference of squares of independent linear differential operators mapping $\mathfrak{C}^\infty(\mathbb{R}, \mathbb{R}^{\mathtt{w}})$ into $\mathfrak{C}^\infty(\mathbb{R}, \mathbb{R})$, i.e., with the rows of $\mathrm{col}(F_+, F_-)$ linearly independent over $\mathbb{R}$. The signature of QDFs play an important role in the proofs.

A QDF restricted to a behavior can also be viewed as a QF on $\mathfrak{B}$, by the map $w \in \mathfrak{B} \mapsto Q_\Phi(w)(0)$. We denote it as $Q_\Phi|_\mathfrak{B}$, its rank as $\mathrm{rank}(Q_\Phi|_\mathfrak{B})$, and its signature as $\mathrm{sign}(Q_\Phi|_\mathfrak{B}) = (\sigma_-(Q_\Phi|_\mathfrak{B}), \sigma_+(Q_\Phi|_\mathfrak{B}))$. For $\mathfrak{B} \in \mathfrak{L}^{\mathtt{w}}_{\mathrm{cont}}$, these can be computed as follows. Let $w = M(d/dt)\ell$ be an observable image representation of $\mathfrak{B}$. Define Φ' as $\Phi'(\zeta, \eta) = M^T(\zeta)\Phi(\zeta, \eta)M(\eta)$. Then the rank and signature $Q_\Phi|_\mathfrak{B}$ equal those of $Q_{\Phi'}$. The signature of $Q_\Phi|_\mathfrak{B}$ has the following significance. There exist polynomial matrices $F^+_\Phi \in \mathbb{R}^{\sigma_+(Q_\Phi|_\mathfrak{B})\times\mathtt{w}}[\xi]$, $F^-_\Phi \in \mathbb{R}^{\sigma_-(Q_\Phi|_\mathfrak{B})\times\mathtt{w}}[\xi]$ such that $Q_\Phi(w)$ can be factored as $Q_\Phi(w) = |F^+_\Phi(d/dt)w|^2 - |F^-_\Phi(d/dt)w|^2$ for $w \in \mathfrak{B}$. We call such a factorization of $Q_\Phi|_\mathfrak{B}$ *canonical*. Any other factorization $Q_\Phi(w) = |F^+(d/dt)w|^2 - |F^-(d/dt)w|^2$ for $w \in \mathfrak{B}$ satisfies $\mathrm{rowdim}(F^+) \geq \sigma_+(Q_\Phi|_\mathfrak{B})$ and $\mathrm{rowdim}(F^-) \geq \sigma_-(Q_\Phi|_\mathfrak{B})$. A canonical factorization of $Q_\Phi|_\mathfrak{B}$ can be obtained from an observable image representation $w = M(d/dt)\ell$ of $\mathfrak{B}$. First obtain a canonical factorization of $Q_{\Phi'}$ on $\mathfrak{C}^\infty(\mathbb{R}, \mathbb{R}^{\mathtt{w}})$, by factoring the $\mathrm{mat}(\Phi')$, yielding $Q_{\Phi'}(\ell) = |F^+_{\Phi'}(d/dt)\ell|^2 - |F^-_{\Phi'}(d/dt)\ell|^2$, and express ℓ in terms of w, using $\ell = M^L(d/dt)w$, with M^L a polynomial matrix left inverse of M.

VII. Proofs of the Main Results

The proofs are organized as follows. The proofs of the main results are interlaced with propositions and lemmas of peripheral interest. The proofs of these results are given in Section VIII, together with the proofs of the propositions and lemmas in the main text.

A. Proof of Theorem 1

(*Only if*): Let $\mathcal{P}_{\mathrm{full}} \in \mathfrak{L}^{\mathtt{v}+\mathtt{c}}$ be the full behavior of the plant. Assume that $\mathcal{K} \in \mathfrak{L}^{\mathtt{v}}$ is implemented by $\mathcal{C} \in \mathfrak{L}^{\mathtt{c}}$. Then

$$\begin{aligned}\mathcal{P} &= \{v \in \mathfrak{C}^\infty(\mathbb{R}, \mathbb{R}^{\mathtt{v}}) \mid \exists\, c \in \mathfrak{C}^\infty(\mathbb{R}, \mathbb{R}^{\mathtt{c}}) \\ &\qquad \text{such that } (v, c) \in \mathcal{P}_{\mathrm{full}}\} \\ \mathcal{K} &= \{v \in \mathfrak{C}^\infty(\mathbb{R}, \mathbb{R}^{\mathtt{v}}) \mid \exists\, c \in \mathcal{C} \text{ such that } (v, c) \in \mathcal{P}_{\mathrm{full}}\} \\ \mathcal{N} &= \{v \in \mathfrak{C}^\infty(\mathbb{R}, \mathbb{R}^{\mathtt{v}}) \mid (v, 0) \in \mathcal{P}_{\mathrm{full}}\}.\end{aligned}$$

Clearly, $\mathcal{N} \subset \mathcal{K} \subset \mathcal{P}$, as claimed.

(*If*): This part uses kernel representations. We need the following standard result.

Lemma 14: Let $\mathfrak{B}, \mathfrak{B}' \in \mathfrak{L}^{\mathtt{w}}$, with kernel representations $L(d/dt)w = 0$ and $L'(d/dt)w = 0$, respectively. Then $\mathfrak{B}' \subset \mathfrak{B}$ if and only if there exists $F \in \mathbb{R}^{\bullet\times\bullet}[\xi]$ such that $L = FL'$.

Assume that $\mathcal{N} \subset \mathcal{K} \subset \mathcal{P}$. Let $R(d/dt)v = M(d/dt)c$ be a kernel representation of $\mathcal{P}_{\mathrm{full}}$. Then $R(d/dt)v = 0$ is a kernel representation of $\mathcal{N}$. Using the lemma, we conclude that $\mathcal{K}$ admits a kernel representation of the form $F(d/dt)R(d/dt)v = 0$ for some $F \in \mathbb{R}^{\bullet\times\bullet}[\xi]$. We now prove that the controller $F(d/dt)M(d/dt)c = 0$ implements $\mathcal{K}$, in other words, that $\mathcal{K}'$, the manifest behavior of $R(d/dt)v = M(d/dt)c$, $F(d/dt)M(d/dt)c = 0$, equals $\mathcal{K}$.

Let $v' \in \mathcal{K}'$. Then v' satisfies $R(d/dt)v' = M(d/dt)c$ for some c such that $F(d/dt)M(d/dt)c = 0$. This implies $F(d/dt)R(d/dt)v' = 0$, whence $v' \in \mathcal{K}$, and $\mathcal{K}' \subset \mathcal{K}$. Conversely, let $v \in \mathcal{K}$. Then $F(d/dt)R(d/dt)v = 0$, and there is a c such that $R(d/dt)v = M(d/dt)c$. This c hence satisfies also $F(d/dt)M(d/dt)c = 0$. Whence v is such that $R(d/dt)v = M(d/dt)c$ for some c that satisfies $F(d/dt)M(d/dt)c = 0$. Consequently, $v \in \mathcal{K}'$, and $\mathcal{K} \subset \mathcal{K}'$.

If $\mathcal{K} \in \mathfrak{L}^{\mathtt{v}}_{\mathrm{cont}}$, then the controller that implements it can also be taken to be controllable. For if $\mathcal{C}$ implements $\mathcal{K}$, then its controllable part also implements $\mathcal{K}$.

B. Proof of Theorem 5

1) Proof of Theorem 5, "Only If"-Part: The key to the "only if"-part is the matching of a system with its orthogonal complement. Assume that $\mathcal{K} \in \mathfrak{L}^{\mathtt{v}}_{\mathrm{cont}}$ satisfies $\mathcal{N} \subset \mathcal{K} \subset \mathcal{P}$, is Σ-dissipative on $\mathbb{R}_-$, and $\mathtt{m}(\mathcal{K}) = \sigma_+(\Sigma)$. Since $\mathcal{K} \supset \mathcal{N}$, and $\mathcal{K}$ is Σ-dissipative, so is $\mathcal{N}$. Since $\mathcal{K}$ is Σ-dissipative, $\mathcal{K}^{\perp_\Sigma}$ is $(-\Sigma)$-dissipative by proposition 12. Furthermore, $\mathcal{K} \subset \mathcal{P}$. Therefore $\mathcal{P}^{\perp_\Sigma} \subset \mathcal{K}^{\perp_\Sigma}$, hence $\mathcal{P}^{\perp_\Sigma}$ is also $(-\Sigma)$-dissipative. Since $\mathcal{K}$ is Σ-dissipative on $\mathbb{R}_-$ and $\mathtt{m}(\mathcal{K}) = \sigma_+(\Sigma)$, every one of its storage functions is, by proposition 12, of the form $|X_\mathcal{K}(d/dt)v_1|^2_K$, $v_1 \in \mathcal{K}$, with $X_\mathcal{K}(d/dt)$ a minimal state map for $\mathcal{K}$ and $K = K^T > 0$.

Let $(X_\mathcal{K}(d/dt), Z_\mathcal{K}(d/dt))$ be a matched pair of minimal state maps for $\mathcal{K}$ and $\mathcal{K}^\perp$. Then $(|X_\mathcal{K}(d/dt)v_1|^2_K, -|Z_\mathcal{K}(d/dt)\Sigma v_2|^2_{K^{-1}})$ is a Σ-matched pair of storage functions for $\mathcal{K}$ and $\mathcal{K}^{\perp_\Sigma}$ as, respectively, Σ- and $(-\Sigma)$-dissipative systems. Consequently

$$\begin{aligned}\frac{d}{dt}\left|X_\mathcal{K}\left(\frac{d}{dt}\right)v_1\right|^2_K &\leq |v_1|^2_\Sigma \\ -\frac{d}{dt}\left|Z_\mathcal{K}\left(\frac{d}{dt}\right)\Sigma v_2\right|^2_{K^{-1}} &\leq -|v_2|^2_\Sigma \\ \frac{d}{dt}\left(X_\mathcal{K}\left(\frac{d}{dt}\right)v_1\right)^T\left(Z_\mathcal{K}\left(\frac{d}{dt}\right)\Sigma v_2\right) &= v_1^T\Sigma v_2\end{aligned}$$

for all $v_1 \in \mathcal{K}$ and $v_2 \in \mathcal{K}^{\perp_\Sigma}$. Since $\mathcal{N} \subset \mathcal{K} \subset \mathcal{P}$, whence $\mathcal{P}^{\perp_\Sigma} \subset \mathcal{K}^{\perp_\Sigma}$, these relations also hold for $v_1 \in \mathcal{N}$ and $v_2 \in \mathcal{P}^{\perp_\Sigma}$. This implies

$$\begin{aligned}&\left|X_\mathcal{K}\left(\frac{d}{dt}\right)v_1\right|^2_K + \left|Z_\mathcal{K}\left(\frac{d}{dt}\right)\Sigma v_2\right|^2_{K^{-1}} \\ &\quad + 2\left(X_\mathcal{K}\left(\frac{d}{dt}\right)v_1\right)^T\left(Z_\mathcal{K}\left(\frac{d}{dt}\right)\Sigma v_2\right) \\ &= \left|X_\mathcal{K}\left(\frac{d}{dt}\right)v_1 + K^{-1}Z_\mathcal{K}\left(\frac{d}{dt}\right)\Sigma v_2\right|^2_K.\end{aligned}$$

for $v_1 \in \mathcal{N}$ and $v_2 \in \mathcal{P}^{\perp_\Sigma}$. Since K is positive definite, the above QDF is nonnegative. Now identify $Q_{\Psi_\mathcal{N}}(v_1) = |X_\mathcal{K}(d/dt)v_1|^2_K$, $Q_{\Psi_{\mathcal{P}^{\perp_\Sigma}}}(v_2) = -|Z_\mathcal{K}(d/dt)\Sigma v_2|^2_{K^{-1}}$, and $L_{\Psi_{(\mathcal{N}, \mathcal{P}^{\perp_\Sigma})}}(v_1, v_2) = (X_\mathcal{K}(d/dt)v_1)^T(Z_\mathcal{K}(d/dt)\Sigma v_2)$ as

QDF's and a BDF that satisfy the conditions required by theorem 5. This proves the "only if"-part of Theorem 5.

2) Proof of Theorem 5, "If"-Part: We first explain, very intuitively and informally, the idea behind the construction of the desired behavior $\mathcal{K} \in \mathfrak{L}^{\mathtt{v}}$. Consider the QDF $|v|_\Sigma^2$ for $v \in \mathfrak{C}^\infty(\mathbb{R}, \mathbb{R}^{\mathtt{v}})$. Somehow, we must decompose $\mathfrak{C}^\infty(\mathbb{R}, \mathbb{R}^{\mathtt{v}})$ into $\mathcal{K}+\mathcal{K}^{\perp_\Sigma}$ (in this intuitive explanation, we assume the behaviors and their Σ-complement are complementary) such that $\mathcal{N} \subset \mathcal{K} \subset \mathcal{P}$, $\mathcal{K}$ is Σ-dissipative on $\mathbb{R}_-$, and $\mathtt{m}(\mathcal{K}) = \sigma_+(\mathcal{K})$. Decompose $\mathfrak{C}^\infty(\mathbb{R}, \mathbb{R}^{\mathtt{v}})$ as $\mathcal{N} + \mathcal{P}^{\perp_\Sigma} + \mathcal{P} \cap \mathcal{N}^{\perp_\Sigma}$. Because of the requirements, it is logical to look for a $\mathcal{K}$ that is the sum of $\mathcal{N}$ and a sub-behavior of $\mathcal{P} \cap \mathcal{N}^{\perp_\Sigma}$. This behavior has then no intersection with $\mathcal{P}^{\perp_\Sigma}$, which we have to avoid, since it is $-\Sigma$ dissipative by assumption. So, we will look for a subspace of $\mathcal{P} \cap \mathcal{N}^{\perp_\Sigma}$ that is Σ dissipative. Clearly this yield a behavior $\mathcal{K}$ such that $\mathcal{N} \subset \mathcal{K} \subset \mathcal{P}$, and that is Σ-dissipative. By exploiting the coupling condition, we deduce that our choice is actually Σ-dissipative on $\mathbb{R}_-$. The hard part, which requires very delicate estimates, will be to prove that $\mathtt{m}(\mathcal{K}) = \sigma_+(\mathcal{K})$.

Consider the "coupling" QDF $Q_{\Psi\mathcal{N}}(v_1) - Q_{\Psi_{\mathcal{P}^{\perp_\Sigma}}}(v_2) + 2L_{\Psi_{(\mathcal{N},\,\mathcal{P}^{\perp_\Sigma})}}(v_1, v_2)$ for $(v_1, v_2) \in \mathcal{N} \times \mathcal{P}^{\perp_\Sigma}$. Let $(X_\mathcal{N}, Z_\mathcal{N})$ and $(X_\mathcal{P}, Z_\mathcal{P})$ be matched pairs of minimal state maps for, respectively, $(\mathcal{N}, \mathcal{N}^\perp)$ and $(\mathcal{P}, \mathcal{P}^\perp)$. By Proposition 10 and corollary 11, there exist matrices $K_\mathcal{N} = K_\mathcal{N}^T \in \mathbb{R}^{\mathtt{n}(\mathcal{N})\times\mathtt{n}(\mathcal{N})}$, $K_{\mathcal{P}^{\perp_\Sigma}} = K_{\mathcal{P}^{\perp_\Sigma}}^T \in \mathbb{R}^{\mathtt{n}(\mathcal{P})\times\mathtt{n}(\mathcal{P})}$, and $K_{(\mathcal{N},\,\mathcal{P}^\perp)} \in \mathbb{R}^{\mathtt{n}(\mathcal{N})\times\mathtt{n}(\mathcal{P})}$, such that

$$\begin{aligned} Q_{\Psi_N}(v_1) &= \left|X_\mathcal{N}\left(\frac{d}{dt}\right)v_1\right|_{K_\mathcal{N}}^2 \\ Q_{\Psi^{\perp_\Sigma}}(v_2) &= \left|Z_\mathcal{P}\left(\frac{d}{dt}\right)\Sigma v_2\right|_{K_{\mathcal{P}^{\perp_\Sigma}}}^2 \\ L_{\Psi_{\mathcal{N},\,\mathcal{P}^{\perp_\Sigma}}}(v_1, v_2) &= \left(X_\mathcal{N}\left(\frac{d}{dt}\right)v_1\right)^T K_{(\mathcal{N},\,\mathcal{P}^\perp)} \\ &\quad \times \left(Z_\mathcal{P}\left(\frac{d}{dt}\right)\Sigma v_2\right) \end{aligned}$$

for $(v_1, v_2) \in \mathcal{N} \times \mathcal{P}^{\perp_\Sigma}$. Since $X_\mathcal{N}$ and $Z_\mathcal{P}$ are minimal, hence trim, state maps for respectively $\mathcal{N}$ and $\mathcal{P}^\perp$, the maps $v_1 \in \mathcal{N} \mapsto X_\mathcal{N}(d/dt)v_1(0) \in \mathbb{R}^{\mathtt{n}(\mathcal{N})}$ and $v_2 \in \mathcal{P}^{\perp_\Sigma} \mapsto Z_\mathcal{P}(d/dt)\Sigma v_2(0) \in \mathbb{R}^{\mathtt{n}(\mathcal{P})}$ are surjective. The coupling condition (1) therefore implies that

$$\begin{aligned} K = K^T &= \begin{bmatrix} K_\mathcal{N} & K_{(\mathcal{N},\,\mathcal{P}^\perp)} \\ K_{(\mathcal{N},\,\mathcal{P}^\perp)} & -K_{\mathcal{P}^{\perp_\Sigma}} \end{bmatrix} \\ &\in \mathbb{R}^{\mathtt{n}(\mathcal{N})+\mathtt{n}(\mathcal{P}))\times(\mathtt{n}(\mathcal{N})+\mathtt{n}(\mathcal{P}))} \end{aligned} \tag{3}$$

is a nonnegative definite matrix. Assume, in order to explain the intuitive idea, that it is actually positive–definite: $K = K^T > 0$. Consider the QDFs

$$\begin{aligned} Q_\Delta(v_1, v_2) &= |v_1 + v_2|_\Sigma^2 \\ &\quad - \frac{d}{dt}\left|\mathrm{col}\left(X_\mathcal{N}\left(\frac{d}{dt}\right)v_1, Z_\mathcal{P}\left(\frac{d}{dt}\right)\Sigma v_2\right)\right|_K^2 \end{aligned}$$

for $(v_1, v_2) \in \mathcal{N} \times \mathcal{P}^{\perp_\Sigma}$, and

$$\begin{aligned} Q_\Psi(v_3) &= |v_3|_\Sigma^2 \\ &\quad - \frac{d}{dt}\left|\mathrm{col}\left(Z_\mathcal{N}\left(\frac{d}{dt}\right)\Sigma v_3, X_\mathcal{P}\left(\frac{d}{dt}\right)v_3\right)\right|_{K^{-1}}^2 \end{aligned} \tag{4}$$

for $v_3 \in \mathcal{P} \cap \mathcal{N}^{\perp_\Sigma}$. The definition of $L_{\Psi_{(\mathcal{N},\,\mathcal{P}^{\perp_\Sigma})}}$ yields the decomposition

$$\begin{aligned} Q_\Delta(v_1, v_2) &= |v_1|_\Sigma^2 - \frac{d}{dt}\left|X_\mathcal{N}\left(\frac{d}{dt}\right)v_1\right|_{K_\mathcal{N}}^2 \\ &\quad + |v_2|_\Sigma^2 + \frac{d}{dt}\left|Z_\mathcal{P}\left(\frac{d}{dt}\right)\Sigma v_2\right|_{K_{\mathcal{P}^{\perp_\Sigma}}}^2 \\ &= Q_\Delta(v_1, 0) + Q_\Delta(0, v_2) \end{aligned} \tag{5}$$

for $v_1 \in \mathcal{N}$ and $v_2 \in \mathcal{P}^{\perp_\Sigma}$. Factor the QDF $Q_\Psi|_{\mathcal{P}\cap\mathcal{N}^{\perp_\Sigma}}$ canonically as

$$Q_\Psi(v_3) = \left|F^+\left(\frac{d}{dt}\right)v_3\right|^2 - \left|F^-\left(\frac{d}{dt}\right)v_3\right|^2. \tag{6}$$

This yields the following crucial equality:

$$\begin{aligned} &\frac{d}{dt}\left|\mathrm{col}\left(X_\mathcal{N}\left(\frac{d}{dt}\right)v_1,\, Z_\mathcal{P}\left(\frac{d}{dt}\right)\Sigma v_2\right)\right. \\ &\qquad \left. +K^{-1}\mathrm{col}\left(Z_\mathcal{N}\left(\frac{d}{dt}\right)\Sigma v_3,\, X_\mathcal{P}\left(\frac{d}{dt}\right)v_3\right)\right|_K^2 \\ &\quad = |v_1 + v_2 + v_3|_\Sigma^2 - Q_\Delta(v_1, v_2) - Q_\Psi(v_3) \\ &\quad = |v_1 + v_2 + v_3|_\Sigma^2 - Q_\Delta(v_1, 0) - Q_\Delta(0, v_2) - Q_\Psi(v_3) \\ &\quad = |v_1 + v_2 + v_3|_\Sigma^2 - Q_\Delta(v_1, 0) - Q_\Delta(0, v_2) \\ &\qquad - \left|F^+\left(\frac{d}{dt}\right)v_3\right|^2 + \left|F^-\left(\frac{d}{dt}\right)v_3\right|^2 \end{aligned} \tag{7}$$

for $v_1 \in \mathcal{N}$, $v_2 \in \mathcal{P}^{\perp_\Sigma}$, and $v_3 \in \mathcal{P} \cap \mathcal{N}^{\perp_\Sigma}$. Assume also, in order to explain the idea, that $\mathcal{N} + \mathcal{P}^{\perp_\Sigma} + \mathcal{P} \cap \mathcal{N}^{\perp_\Sigma} = \mathfrak{C}^\infty(\mathbb{R}, \mathbb{R}^{\mathtt{v}})$. Equation (7) then yields a very transparent decomposition of any $v \in \mathfrak{C}^\infty(\mathbb{R}, \mathbb{R}^{\mathtt{v}})$ as $v = v_1 + v_2 + v_3$ that nicely puts the sign of $|v|_\Sigma^2$ in evidence. It is this decomposition that we are after. Examine the signs of terms on the right-hand side of (7). Since $\mathcal{N}$ is Σ-dissipative, with $|X_\mathcal{N}(d/dt)v_1|_{K_\mathcal{N}}^2$ as storage function, and $\mathcal{P}^{\perp_\Sigma}$ is $(-\Sigma)$-dissipative, with $|Z_\mathcal{P}(d/dt)\Sigma v_2|_{K_{\mathcal{P}^{\perp_\Sigma}}}^2$ as storage function, $Q_\Delta(v_1, 0) \geq 0$ for $v_1 \in \mathcal{N}$, and $Q_\Delta(0, v_2) \leq 0$ for $v_2 \in \mathcal{P}^{\perp_\Sigma}$. The idea behind the construction of the controlled behavior $\mathcal{K}$ is to cancel the two nonnegative terms $-Q_\Delta(0, v_2) + |F^-(d/dt)v_3|^2$ on the right hand side of (7), by taking for $\mathcal{K}$

$$\boxed{\mathcal{K} = \mathcal{N} + \mathcal{F}^-} \tag{8}$$

with $\mathcal{F}^-$ the controllable part of the behavior $\{v_3 \in \mathcal{P} \cap \mathcal{N}^{\perp_\Sigma} | F^-(d/dt)v_3 = 0\}$. Equation (7) then yields

$$\begin{aligned} &\frac{d}{dt}\left|\mathrm{col}\left(X_\mathcal{N}\left(\frac{d}{dt}\right)v_1,\, 0\right)\right. \\ &\qquad \left. +K^{-1}\mathrm{col}\left(Z_\mathcal{N}\left(\frac{d}{dt}\right)\Sigma v_3,\, X_\mathcal{P}\left(\frac{d}{dt}\right)v_3\right)\right|_K^2 \\ &\quad = |v_1 + v_3|_\Sigma^2 - Q_\Delta(v_1, 0) - \left|F^+\left(\frac{d}{dt}\right)v_3\right|^2 \\ &\quad \leq |v_1 + v_3|_\Sigma^2. \end{aligned} \tag{9}$$

This shows that $\mathcal{K}$ defined by (8) is indeed Σ-dissipative on $\mathbb{R}_-$.

The difficult part is to show that $\mathtt{m}(\mathcal{K}) = \sigma_+(\Sigma)$. For this to be the case, $\mathtt{m}(\mathcal{F}^-)$ should not be too low, in other words, the row dimension of F^- should not be too high. Obtaining a sharp estimate of this row dimension requires a sharp estimate of the signature of $Q_\Psi|_{\mathcal{P}\cap\mathcal{N}^{\perp_\Sigma}}$. This forms the *pièce de résistance* of the proof.

Our strategy for the formal proof is as follows. First, we give the proof under certain regularity assumptions and subsequently we consider the general case.

3) The Regular Case: In the *regular case* we assume that i) $K = K^T > 0$ (as in the above explanation of the construction of $\mathcal{K}$) and ii) $\mathcal{N} + \mathcal{N}^{\perp_\Sigma} = \mathfrak{C}^\infty(\mathbb{R}, \mathbb{R}^{\mathtt{v}})$ (equivalently, by proposition 9, ii)' $\mathtt{m}(\mathcal{N}\cap\mathcal{N}^{\perp_\Sigma}) = 0$). The origin of the second condition may be understood as follows. When $\mathcal{N}$ is Σ-dissipative, then $\|v\|^2_\Sigma \ge 0$ for all $v \in \mathcal{N}\cap\mathfrak{D}(\mathbb{R}, \mathbb{R}^{\mathtt{v}})$. The stronger condition $\|v\|^2_\Sigma > 0$ for all $0 \neq v \in \mathcal{N}\cap\mathfrak{D}(\mathbb{R}, \mathbb{R}^{\mathtt{v}})$ implies $\mathcal{N} + \mathcal{N}^{\perp_\Sigma} = \mathfrak{C}^\infty(\mathbb{R}, \mathbb{R}^{\mathtt{v}})$. For otherwise, there exists $0 \neq v \in \mathcal{N}\cap\mathcal{N}^{\perp_\Sigma}\cap\mathfrak{D}(\mathbb{R}, \mathbb{R}^{\mathtt{v}})$ yielding $\|v\|^2_\Sigma = 0$. The second regularity condition may thus be viewed as a form of strict Σ-dissipativity of $\mathcal{N}$.

We now derive the following relation between the signatures and ranks of the QDF's defined above:

$$\begin{aligned}\sigma_-\left(Q_\Psi|_{\mathcal{P}\cap\mathcal{N}^{\perp_\Sigma}}\right) \le{}& \operatorname{rank}(Q_\Delta|_{\mathcal{N}\times 0}) \\ &+ \sigma_-(\Sigma) - \mathtt{m}(\mathcal{N}) - \mathtt{p}(\mathcal{P}) \\ \sigma_+\left(Q_\Psi|_{\mathcal{P}\cap\mathcal{N}^{\perp_\Sigma}}\right) \le{}& \operatorname{rank}(Q_\Delta|_{0\times\mathcal{P}^{\perp_\Sigma}}) \\ &+ \sigma_+(\Sigma) - \mathtt{m}(\mathcal{N}) - \mathtt{p}(\mathcal{P}).\end{aligned} \tag{10}$$

The proof of these relations involves five steps.

1) Consider the following subspaces $\mathbb{L}_\Delta$ and $\mathbb{L}_\Psi$ of $\mathbb{R}^{\mathtt{v}+2(\mathtt{n}(\mathcal{N})+\mathtt{n}(\mathcal{P}))}$:

$$\begin{aligned}\mathbb{L}_\Delta = \Bigg\{ a | \exists\, v_1 \in \mathcal{N} \text{ and } v_2 \in \mathcal{P}^{\perp_\Sigma} \text{ such that} \\ a = \left(\operatorname{col}\left(v_1 + v_2, X_\mathcal{N}\left(\frac{d}{dt}\right)v_1, Z_\mathcal{P}\left(\frac{d}{dt}\right)\Sigma v_2,\right.\right. \\ \left.\left.\frac{d}{dt}X_\mathcal{N}\left(\frac{d}{dt}\right)v_1, \frac{d}{dt}Z_\mathcal{P}\left(\frac{d}{dt}\right)\Sigma v_2\right)\right)(0)\Bigg\}\end{aligned}$$

$$\begin{aligned}\mathbb{L}_\Psi = \Bigg\{ b | \exists\, v_3 \in \mathcal{P}\cap\mathcal{N}^{\perp_\Sigma} \text{ such that} \\ b = \left(\operatorname{col}\left(\Sigma v_3, -\frac{d}{dt}Z_\mathcal{N}\left(\frac{d}{dt}\right)\Sigma v_3,\right.\right. \\ -\frac{d}{dt}X_\mathcal{P}\left(\frac{d}{dt}\right)v_3, -Z_\mathcal{N}\left(\frac{d}{dt}\right)\Sigma v_3 \\ \left.\left.-X_\mathcal{P}\left(\frac{d}{dt}\right)v_3\right)\right)(0)\Bigg\}.\end{aligned}$$

Observe the following orthogonality relations:

$$\begin{aligned}\frac{d}{dt}\left(X_\mathcal{N}\left(\frac{d}{dt}\right)v_1\right)^T\left(Z_\mathcal{N}\left(\frac{d}{dt}\right)\Sigma v_3\right) &= v_1^T\Sigma v_3 \\ \frac{d}{dt}\left(X_\mathcal{P}\left(\frac{d}{dt}\right)v_3\right)^T\left(Z_\mathcal{P}\left(\frac{d}{dt}\right)\Sigma v_2\right) &= v_3^T\Sigma v_2\end{aligned}$$

for $v_1 \in \mathcal{N}$, $v_2 \in \mathcal{P}^{\perp_\Sigma} \subset \mathcal{N}^{\perp_\Sigma}$, and $v_3 \in \mathcal{P}\cap\mathcal{N}^{\perp_\Sigma}$, hence $\Sigma v_2 \in \mathcal{N}$. These identities imply that $\mathbb{L}_\Psi \subset \mathbb{L}_\Delta^\perp$. Consider also the QF on $\mathbb{R}^{\mathtt{v}+2(\mathtt{n}(\mathcal{N})+\mathtt{n}(\mathcal{P}))}$ induced by

$$Q = \begin{bmatrix}\Sigma & 0 & 0 \\ 0 & 0 & -K \\ 0 & -K & 0\end{bmatrix}.$$

Note that, since Σ and K are invertible, so is Q. Since $\mathbb{L}_\Psi \subset \mathbb{L}_\Delta^\perp$, we obtain

$$\begin{aligned}\sigma_-\left(Q^{-1}|_{\mathbb{L}_\Psi}\right) &\le \sigma_-\left(Q|_{(Q\mathbb{L}_\Delta)^\perp}\right), \\ \sigma_+\left(Q^{-1}|_{\mathbb{L}_\Psi}\right) &\le \sigma_+\left(Q|_{(Q\mathbb{L}_\Delta)^\perp}\right).\end{aligned} \tag{11}$$

2) Note that $\operatorname{sign}(Q) = \operatorname{sign}(\Sigma) + (\operatorname{rank}(K), \operatorname{rank}(K))$, and $\operatorname{rank}(K)) = \mathtt{n}(\mathcal{N}) + \mathtt{n}(\mathcal{P})$. Lemma 13 and $\ker(Q) = 0$ imply

$$\begin{aligned}\sigma_-\left(Q|_{(Q\mathbb{L}_\Delta)^\perp}\right) ={}& \sigma_+(Q|_{\mathbb{L}_\Delta}) + \sigma_-(\Sigma) + \mathtt{n}(\mathcal{N}) \\ &+ \mathtt{n}(\mathcal{P}) - \dim(\mathbb{L}_\Delta) \\ \sigma_+\left(Q|_{(Q\mathbb{L}_\Delta)^\perp}\right) ={}& \sigma_-(Q|_{\mathbb{L}_\Delta}) + \sigma_+(\Sigma) \\ &+ \mathtt{n}(\mathcal{N}) + \mathtt{n}(\mathcal{P}) - \dim(\mathbb{L}_\Delta).\end{aligned} \tag{12}$$

3) Next, calculate $\dim(\mathbb{L}_\Delta)$, using the lemma below. To see that the dimension count in the lemma is reasonable, let $\mathfrak{B} \in \mathfrak{L}^\bullet$. With $X(d/dt)$ a minimal state map for $\mathfrak{B}$, define $\mathbb{L}_\mathfrak{B} = \{a \in \mathbb{R}^{\mathtt{m}(\mathfrak{B})+2\mathtt{n}(\mathfrak{B})} | \exists\, w \in \mathfrak{B}$ such that $a = (w, X(d/dt)w, (d/dt)X(d/dt)w)(0)\}$. It is easy to see that $\dim(\mathbb{L}_\mathfrak{B}) = \mathtt{m}(\mathfrak{B}) + \mathtt{n}(\mathfrak{B})$. This suggests $\dim(\mathbb{L}_\Delta) = \mathtt{m}(\mathcal{N}) + \mathtt{p}(\mathcal{P}) + \mathtt{n}(\mathcal{N}) + \mathtt{n}(\mathcal{P})$, but this count is too rough, since the combination of $v_1 \in \mathcal{N}$ and $v_2 \in \mathcal{P}^{\perp_\Sigma}$ into $v_1 + v_2$ in the definition of $\mathbb{L}_\Delta$ may absorb degrees of freedom. For the exact count, we need the memoryless part of a behavior, introduced in Section VI-A.

Lemma 15: The dimension of $\mathbb{L}_\Delta$ is given by

$$\begin{aligned}\dim(\mathbb{L}_\Delta) ={}& \mathtt{m}(\mathcal{N}) + \mathtt{p}(\mathcal{P}) + \mathtt{n}(\mathcal{N}) + \mathtt{n}(\mathcal{P}) \\ &- \mathtt{m}(\mathcal{N}_{\text{memoryless}} \cap (\mathcal{P}^{\perp_\Sigma})_{\text{memoryless}}).\end{aligned}$$

Hence, in the regular case, $\dim(\mathbb{L}_\Delta) = \mathtt{m}(\mathcal{N}) + \mathtt{p}(\mathcal{P}) + \mathtt{n}(\mathcal{N}) + \mathtt{n}(\mathcal{P})$.

Using this lemma in (12), combined with (11), yields

$$\begin{aligned}\sigma_-\left(Q^{-1}|_{\mathbb{L}_\Psi}\right) &\le \sigma_+(Q|_{\mathbb{L}_\Delta}) + \sigma_-(\Sigma) - \mathtt{m}(\mathcal{N}) - \mathtt{p}(\mathcal{P}) \\ \sigma_+\left(Q^{-1}|_{\mathbb{L}_\Psi}\right) &\le \sigma_-(Q|_{\mathbb{L}_\Delta}) + \sigma_+(\Sigma) - \mathtt{m}(\mathcal{N}) - \mathtt{p}(\mathcal{P}).\end{aligned} \tag{13}$$

4) The next step consists of observing the equalities

$$\begin{aligned}\operatorname{sign}\left(Q_\Delta|_{\mathcal{N}\times\mathcal{P}^{\perp_\Sigma}}\right) &= \operatorname{sign}(Q|_{\mathbb{L}_\Delta}) \\ \operatorname{sign}(Q_\Psi|_{\mathcal{P}\cap\mathcal{N}^{\perp_\Sigma}}) &= \operatorname{sign}\left(Q^{-1}|_{\mathbb{L}_\Psi}\right).\end{aligned}$$

These follow immediately from the definitions of the various QFs.

5) The final step involves the analysis of the QDF $Q_\Delta|_{\mathcal{N}\times\mathcal{P}^{\perp_\Sigma}}$. Equation (5) yields

$$\operatorname{sign}\left(Q_\Delta|_{\mathcal{N}\times\mathcal{P}^{\perp_\Sigma}}\right) = \left(\operatorname{rank}\left(Q_\Delta|_{0\times\mathcal{P}^{\perp_\Sigma}}\right), \operatorname{rank}\left(Q_\Delta|_{\mathcal{N}\times 0}\right)\right).$$

Now substitute these results into (13). This yields the relations (10).

The next part of the proof requires obtaining a sharp estimate of $\operatorname{rank}(Q_\Delta|_{\mathcal{N}\times 0})$. This, however, requires that the storage function $Q_{\Psi_\mathcal{N}}$ that appears in (1) is an extreme storage function. The following proposition, closely related to Proposition 6, estimates $\operatorname{rank}(Q_\Delta|_{\mathcal{N}\times 0})$ exactly for extreme storage functions.

Proposition 16: Let $\mathfrak{B} \in \mathfrak{L}^{\mathtt{w}}_{\text{cont}}$ and $\Phi = \Phi^T \in \mathbb{R}^{\mathtt{w}\times\mathtt{w}}$ be nonsingular. The ranks of the QDF's $|w|^2_\Phi - (d/dt)Q_{\Psi^{\inf}}(w)$ and $|w|^2_\Phi - (d/dt)Q_{\Psi^{\sup}}(w)$ on $\mathfrak{B}$ are both equal to $\mathtt{m}(\mathfrak{B}) - \mathtt{m}(\mathfrak{B}\cap\mathfrak{B}^{\perp_\Phi})$. In particular, if $\mathfrak{B} + \mathfrak{B}^{\perp_\Phi} = \mathfrak{C}^\infty(\mathbb{R}, \mathbb{R}^{\mathtt{w}})$, then these ranks are equal to $\mathtt{m}(\mathfrak{B})$.

This proposition has the following consequence. By substituting $Q_{\Psi_{\mathcal{N}}}$ by $Q_{\Psi^{\sup}_{\mathcal{N}}}$, the QDF in the statement of theorem 5 yields $\mathrm{rank}(Q_\Delta|_{\mathcal{N}\times 0}) = \mathtt{m}(\mathcal{N}) - \mathtt{m}(\mathcal{N}\cap\mathcal{N}^{\perp_\Sigma})$. Note that the K-matrix, K_{new}, induced by $Q_{\Psi^{\sup}_{\mathcal{N}}}(v_1) + Q_{\Psi_{\mathcal{P}^{\perp_\Sigma}}}(v_2) + L_{\Psi_{\mathcal{N},\mathcal{P}^{\perp_\Sigma}}}(v_1, v_2)$ and the K-matrix, K_{old}, induced by the original $Q_{\Psi_{\mathcal{N}}}(v_1) + Q_{\Psi_{\mathcal{P}^{\perp_\Sigma}}}(v_2) + L_{\Psi_{(\mathcal{N},\mathcal{P}^{\perp_\Sigma})}}(v_1, v_2)$ satisfy $K_{\text{new}} \geq K_{\text{old}}$, since $Q_{\Psi^{\sup}_{\mathcal{N}}}(v_1) + Q_{\Psi_{\mathcal{P}^{\perp_\Sigma}}}(v_2) + L_{\Psi_{(\mathcal{N},\mathcal{P}^{\perp_\Sigma})}}(v_1, v_2) \geq Q_{\Psi_{\mathcal{N}}}(v_1) + Q_{\Psi_{\mathcal{P}^{\perp_\Sigma}}}(v_2) + L_{\Psi_{(\mathcal{N},\mathcal{P}^{\perp_\Sigma})}}(v_1, v_2)$ for $v_1 \in \mathcal{N}$, $v_2 \in \mathcal{P}^{\perp_\Sigma}$. Hence, K_{new} is also positive definite. Regularity implies $\mathtt{m}(\mathcal{N}\cap\mathcal{N}^{\perp_\Sigma}) = 0$. Inequality (10) then yields

$$\sigma_-\left(Q_\Psi|_{\mathcal{P}\cap\mathcal{N}^{\perp_\Sigma}}\right) \leq \sigma_-(\Sigma) - \mathtt{p}(\mathcal{P}) = \mathtt{m}(\mathcal{P}) - \sigma_+(\Sigma) \quad (14)$$

for $v_3 \in \mathcal{P}\cap\mathcal{N}^{\perp_\Sigma}$, with $\mathrm{rowdim}(F^-) = \sigma_-(Q_\Psi|_{\mathcal{P}\cap\mathcal{N}^{\perp_\Sigma}})$.

We prove that $\mathcal{K} \in \mathfrak{L}^{\mathtt{v}}_{\text{cont}}$ as defined by (8) has the following properties, which qualify it as a controlled behavior as required by Theorem 5: i) $\mathcal{N} \subset \mathcal{K} \subset \mathcal{P}$, ii) $\mathcal{K}$ is Σ-dissipative on $\mathbb{R}_-$, and iii) $\mathtt{m}(\mathcal{K}) = \sigma_+(\Sigma)$.

Property i) is obvious. From (9) and $K > 0$, it follows that for every $v_1 \in \mathcal{N}$ and $v_3 \in \mathcal{F}^-$ of compact support there holds $\int_{-\infty}^0 |v_1+v_3|^2_\Sigma\, dt \geq 0$. It is easy to see that any $v \in \mathcal{N}+\mathcal{F}^-$ of compact support may be decomposed as $v_1 + v_3$, with $v_1 \in \mathcal{N}$ and $v_3 \in \mathcal{F}^-$ of compact support. Therefore, $\int_{-\infty}^0 |v|^2_\Sigma\, dt \geq 0$ for all $v \in \mathcal{K}\cap\mathfrak{D}$. This yields ii).

We now show iii). Since $\mathcal{N} + \mathcal{N}^{\perp_\Sigma}\cap\mathcal{P} = \mathcal{P}$, $\mathcal{N} + \mathcal{F}^-$ consists of the signals in $\mathcal{P}$ that satisfy $v_3 \in \mathcal{N}^{\perp_\Sigma}\cap\mathcal{P}$ and $F^-(d/dt)v_3 = 0$. This imposes $\mathtt{p}(\mathcal{P}) + \sigma_-(Q_\Psi|_{\mathcal{P}\cap\mathcal{N}^{\perp_\Sigma}})$ equations. Using (14) indicates that this leaves at least $\sigma_+(\Sigma)$ free components. We now prove this rigorously. We need the following lemma.

Lemma 17: Let $\mathfrak{B} \in \mathfrak{L}^{\mathtt{w}}$ and $F \in \mathbb{R}^{\bullet\times w}[\xi]$. Then the subbehavior $\mathfrak{B}' \subset \mathfrak{B}$ defined by $\mathfrak{B}' = \{w \in \mathfrak{B} | F(d/dt)w = 0\}$ satisfies $\mathtt{m}(\mathfrak{B}') \geq \mathtt{m}(\mathfrak{B}) - \mathrm{rowdim}(F)$.

We are ready to deliver the *coup de grâce*. By the regularity assumption, $\mathtt{m}(\mathcal{N}\cap\mathcal{N}^{\perp_\Sigma}) = 0$, and hence, since $\mathcal{F}^- \subset \mathcal{N}^{\perp_\Sigma}$, $\mathtt{m}(\mathcal{N}\cap\mathcal{F}^-) = 0$. Hence, by Proposition 9, $\mathtt{m}(\mathcal{K}) = \mathtt{m}(\mathcal{N}+\mathcal{F}^-) = \mathtt{m}(\mathcal{N}) + \mathtt{m}(\mathcal{F}^-)$. The above lemma implies $\mathtt{m}(\mathcal{F}^-) \geq \mathtt{m}(\mathcal{P}\cap\mathcal{N}^{\perp_\Sigma}) - \mathrm{rowdim},(F^-)$. This yields $\mathtt{m}(\mathcal{K}) \geq \mathtt{m}(\mathcal{N}) + \mathtt{m}(\mathcal{P}\cap\mathcal{N}^{\perp_\Sigma}) - \mathrm{rowdim}(F^-)$. Combined with (14) and $\mathrm{rowdim}(F^-) = \sigma_-(Q_\Psi|_{\mathcal{P}\cap\mathcal{N}^{\perp_\Sigma}})$, we arrive at $\mathtt{m}(\mathcal{K}) \geq \mathtt{m}(\mathcal{N}) + \mathtt{m}(\mathcal{P}\cap\mathcal{N}^{\perp_\Sigma}) - \sigma_-(\Sigma) + \mathtt{p}(\mathcal{P})$. Further, $\mathcal{N} + (\mathcal{P}\cap\mathcal{N}^{\perp_\Sigma}) = (\mathcal{P}\cap\mathcal{N}) + (\mathcal{P}\cap\mathcal{N}^{\perp_\Sigma})$. Since $\mathcal{N} \subset \mathcal{P}$ and using regularity, this equals $\mathcal{P}\cap(\mathcal{N}+\mathcal{N}^{\perp_\Sigma}) = \mathcal{P}$. Hence by proposition 9, $\mathtt{m}(\mathcal{N}) + \mathtt{m}(\mathcal{P}\cap\mathcal{N}^{\perp_\Sigma}) = \mathtt{m}(\mathcal{P})$. This yields $\mathtt{m}(\mathcal{K}) \geq \mathtt{m}(\mathcal{P}) - \sigma_-(\Sigma) + \mathtt{p}(\mathcal{P})$. Since $\mathtt{m}(\mathcal{P}) + \mathtt{p}(\mathcal{P}) = \mathtt{v}$ and $\sigma_+(\Sigma) = \mathtt{v} - \sigma_-(\Sigma)$, we finally obtain $\mathtt{m}(\mathcal{K}) \geq \sigma_+(\Sigma)$. Hence, by Proposition 2, $\mathtt{m}(\mathcal{K}) = \sigma_+(\Sigma)$.

This completes the proof of the "if"-part of theorem 5 in the regular case. Note that the only point in the proof that the assumption $K > 0$, as opposed to just K nonsingular, is used, is in order to obtain dissipativity on $\mathbb{R}_-$, instead of just on $\mathbb{R}$.

There is an alternative way of defining regularity, namely, by assuming it to mean $K = K^T > 0$ and $\mathcal{P}+\mathcal{P}^{\perp_\Sigma} = \mathfrak{C}^\infty(\mathbb{R}, \mathbb{R}^{\mathtt{v}})$, i.e., strict $(-\Sigma)$-dissipativity of $\mathcal{P}^{\perp_\Sigma}$. This leads to a "dual" way of constructing a controlled behavior, by making in the above proof the substitutions $\mathcal{N} \leftrightarrow \mathcal{P}^{\perp_\Sigma}$, $\mathcal{P} \leftrightarrow \mathcal{N}^{\perp_\Sigma}$, $\Sigma \leftrightarrow -\Sigma$, $\mathbb{R}_- \leftrightarrow \mathbb{R}_+$, and $\mathcal{K} \leftrightarrow \mathcal{K}^{\perp_\Sigma}$.

4) The Singular Case: The main difficulty in the singular case stems from the possible nonsingularity of the matrix K defined by (3). This implies, in particular, that the QDF Q_Ψ given by (4) is not defined. In order to cope with this, we replace the inverse K^{-1} by a symmetric generalized inverse $\mathcal{K}^\#$, i.e., $\mathcal{K}^\# = (\mathcal{K}^\#)^T \in \mathbb{R}^{(\mathtt{n}(\mathcal{N})+\mathtt{n}(\mathcal{P}))\times(\mathtt{n}(\mathcal{N})+\mathtt{n}(\mathcal{P}))}$ satisfies $K = KK^\# K$ and $K^\# = \mathcal{K}^\# K K^\#$, and use the following generalization of the QDF Q_Ψ

$$Q_\Psi(v_3) := |v_3|^2_\Sigma - \frac{d}{dt}\left|\mathrm{col}\left(Z_\mathcal{N}\left(\frac{d}{dt}\right)\Sigma v_3, X_\mathcal{P}\left(\frac{d}{dt}\right)v_3\right)\right|^2_{K^\#}.$$

Obviously, if K is nonsingular this reduces to the old Q_Ψ. Define

$$Q^\# := \begin{bmatrix} \Sigma^{-1} & 0 & 0 \\ 0 & 0 & -K^\# \\ 0 & -K^\# & 0 \end{bmatrix}.$$

We now estimate $\sigma_-(Q|_{(Q\mathbb{L}_\Delta)^\perp})$. Lemma 13 implies

$$\sigma_-\left(Q|_{(Q\mathbb{L}_\Delta)^\perp}\right) = \sigma_+\left(Q|_{\mathbb{L}_\Delta}\right) + \sigma_-(Q) - \dim(\mathbb{L}_\Delta) + \dim(\mathbb{L}_\Delta \cap \ker(Q)). \quad (15)$$

Now compute one by one the terms on the right-hand side of this relation.

1) Since $\sigma_+(Q|_{\mathbb{L}_\Delta}) = \sigma_+(Q_\Delta|_{\mathcal{N}\times\mathcal{P}^{\perp_\Sigma}})$, we obtain, using proposition 16, $\sigma_+(Q|_{\mathbb{L}_\Delta}) = \mathtt{m}(\mathcal{N}) - \mathtt{m}(\mathcal{N}\cap\mathcal{N}^{\perp_\Sigma})$.
2) Obviously, $\sigma_-(Q) = \sigma_-(\Sigma) + \mathrm{rank}(K)$.
3) In order to compute the last two terms, we use convenient representations of $\mathcal{N}$ and $\mathcal{P}$. It turns out that it is easiest to work with a driving variable representation of $\mathcal{N}$ and an output nulling representation of $\mathcal{P}$. Let, therefore

$$\frac{d}{dt}x_\mathcal{N} = A_\mathcal{N} x_\mathcal{N} + B_\mathcal{N} d_\mathcal{N}, \qquad v = C_\mathcal{N} x_\mathcal{N} + D_\mathcal{N} d_\mathcal{N} \quad (16)$$

be a minimal driving variable representation of $\mathcal{N}$, with $x_\mathcal{N} = X_\mathcal{N}(d/dt)v$, $v \in \mathcal{N}$, and $\dim(d_\mathcal{N}) = \mathtt{m}(\mathcal{N})$, and

$$\frac{d}{dt}x_\mathcal{P} = A_\mathcal{P} x_\mathcal{P} + B_\mathcal{P}\Sigma v, \qquad 0 = C_\mathcal{P} x_\mathcal{P} + D_\mathcal{P}\Sigma v \quad (17)$$

be a minimal output nulling representation of $\mathcal{P}$, with $x_\mathcal{P} = X_\mathcal{P}(d/dt)v$, $v \in \mathcal{P}$, and $\mathrm{rowdim}([C_\mathcal{P} \;\; D_\mathcal{P}\Sigma]) = \mathtt{p}(\mathcal{P})$. Using the relations between driving variable and output nulling representations of a system and its orthogonal complement mentioned in Section VI-A, it follows that:

$$\frac{d}{dt}z_\mathcal{P} = -A^T_\mathcal{P} z_\mathcal{P} + C^T_\mathcal{P} d_\mathcal{P}, \qquad v = B^T_\mathcal{P} z_\mathcal{P} - D^T_\mathcal{P} d_\mathcal{P}$$

with $z_\mathcal{P} = Z_\mathcal{P}(d/dt)\Sigma v$, is a driving variable representation of $\mathcal{P}^{\perp_\Sigma}$. Define

$$F = \begin{bmatrix} A_\mathcal{N} & 0 \\ 0 & -A^T_\mathcal{P} \end{bmatrix} \qquad G = \begin{bmatrix} B_\mathcal{N} & 0 \\ 0 & C^T_\mathcal{P} \end{bmatrix}$$
$$H = [\,C_\mathcal{N} \quad B^T_\mathcal{P}\,] \qquad J = [\,D_\mathcal{N} \quad -D^T_\mathcal{P}\,]$$

and $x = (x_{\mathcal{N}}, z_{\mathcal{P}})$, $d = (d_{\mathcal{N}}, d_{\mathcal{P}})$. Then $(d/dt)x = Fx + Gd$, $v = Hx + Jd$ is a driving variable representation of $\mathcal{N} + \mathcal{P}^{\perp_\Sigma}$. This representation allows us to compute $\dim, (\mathbb{L}_\Delta)$ and $\dim(\mathbb{L}_\Delta \cap \ker(Q))$. Note that

$$\mathbb{L}_\Delta = \begin{bmatrix} H & J \\ I & 0 \\ F & G \end{bmatrix}$$

and hence $\dim(\mathbb{L}_\Delta) = \mathtt{n}(\mathcal{N}) + \mathtt{n}(\mathcal{P}) + \mathtt{m}(\mathcal{N}) + \mathtt{p}(\mathcal{P}) - \dim(\ker(\left[\begin{smallmatrix} J \\ G \end{smallmatrix}\right]))$.

4) Next, we compute $\dim(\mathbb{L}_\Delta \cap \ker(Q))$. There holds

$$\begin{aligned}\mathbb{L}_\Delta \cap \ker(Q) &= \Big\{ (v, x, \dot{x}) \in \mathbb{R}^{\mathtt{v}+2(\mathtt{n}(\mathcal{N})+\mathtt{n}(\mathcal{P}))} \Big| \, \Sigma v = 0, \, Kx = 0, \\ &\quad K\dot{x} = 0 \text{ and there exists } d \in \mathbb{R}^{\mathtt{m}(\mathcal{N})+\mathtt{p}(\mathcal{P})} \\ &\quad \text{such that } v = Hx + Jd, \, \dot{x} = Fx + Gd | \Big\}.\end{aligned}$$

Hence

$$\mathbb{L}_\Delta \cap \ker(Q) = \begin{bmatrix} H & J \\ I & 0 \\ F & G \end{bmatrix} \ker(R)$$

with

$$R = \begin{bmatrix} \Sigma H & \Sigma J \\ K & 0 \\ KF & KG \end{bmatrix}. \tag{18}$$

From the general formula $\dim(M\mathfrak{L}) = \dim(\mathfrak{L}) - \dim(\mathfrak{L} \cap \ker(M))$, with M a linear map and $\mathfrak{L}$ a linear subspace of its domain, we, therefore, obtain

$$\dim(\mathbb{L}_\Delta \cap \ker(Q)) = \dim(\ker(R)) - \dim\left(\ker\left(\begin{bmatrix} J \\ G \end{bmatrix}\right)\right).$$

Since the dimension of the kernel of a matrix equals its column dimension minus its rank, this yields

$$\begin{aligned}\dim(\mathbb{L}_\Delta \cap \ker(Q)) &= \mathtt{n}(\mathcal{N}) + \mathtt{n}(\mathcal{P}) + \mathtt{m}(\mathcal{N}) + \mathtt{p}(\mathcal{P}) \\ &\quad - \operatorname{rank}(R) - \dim\left(\ker\left(\begin{bmatrix} J \\ G \end{bmatrix}\right)\right).\end{aligned}$$

Substituting the results of these calculations in (15) yields

$$\begin{aligned}\sigma_-\left(Q|_{(Q\mathbb{L}_\Delta)^\perp}\right) &= \mathtt{m}(\mathcal{N}) - \mathtt{m}(\mathcal{N} \cap \mathcal{N}^{\perp_\Sigma}) \\ &\quad + \sigma_-(\Sigma) + \operatorname{rank}(K) - \operatorname{rank}(R).\end{aligned} \tag{19}$$

With the definition of $\mathbb{L}_\Psi$ we still have, using $\mathbb{L}_\Psi \subset \mathbb{L}_\Delta^\perp$

$$\sigma\left(Q|_{Q^{-1}\mathbb{L}_\Psi}\right) \leq \sigma\left(Q|_{Q^{-1}\mathbb{L}_\Delta^\perp}\right) = \sigma\left(Q|_{(Q\mathbb{L}_\Delta)^\perp}\right)$$

where Q^{-1} denotes the set-theoretic inverse. Of course, we also still have

$$\sigma\left(Q_\Psi|_{\mathcal{P}\cap\mathcal{N}^{\perp_\Sigma}}\right) = \sigma\left(Q^\#|_{\mathbb{L}_\Psi}\right).$$

The difficulty is that in the singular case, we may however not have

$$\sigma(Q^\#|\mathbb{L}_\Psi) = \sigma(Q|Q^{-1}\mathbb{L}_\Psi). \tag{20}$$

In order to circumvent this difficulty, we will restrict the domain of the QDF Q_Ψ to the following sub-behavior of $\mathcal{P} \cap \mathcal{N}^{\perp_\Sigma}$:

$$\begin{aligned}\mathcal{M} := \Bigg\{ v_3 \in \mathcal{P} \cap \mathcal{N}^{\perp_\Sigma} \Bigg| &\operatorname{col}\left(Z_{\mathcal{N}}\left(\frac{d}{dt}\right)\Sigma v_3, \, X_{\mathcal{P}}\left(\frac{d}{dt}\right)v_3\right)(t) \\ &\in \operatorname{im}(K) \, \forall \, t \in \mathbb{R} \Bigg\}.\end{aligned}$$

Now define, in analogy with $\mathbb{L}_\Psi$

$$\begin{aligned}\mathbb{L}'_\Psi := \Bigg\{ b \Bigg| \, &\exists \, v_3 \in \mathcal{M} \text{ such that} \\ &b = \Bigg(\operatorname{col}\Bigg(\Sigma v_3, \, -\frac{d}{dt} Z_{\mathcal{N}}\left(\frac{d}{dt}\right)\Sigma v_3, \, -\frac{d}{dt} X_{\mathcal{P}}\left(\frac{d}{dt}\right) v_3, \\ &\quad -Z_{\mathcal{N}}\left(\frac{d}{dt}\right)\Sigma v_3, \, -X_{\mathcal{P}}\left(\frac{d}{dt}\right) v_3\Bigg)\Bigg)(0)\Bigg\}.\end{aligned}$$

Obviously, $\sigma(Q_\Psi|_{\mathcal{M}}) = \sigma(Q^\#|_{\mathbb{L}'_\Psi})$ and $\mathbb{L}'_\Psi \subset \mathbb{L}_\Psi \cap \operatorname{im}(Q)$. The following lemma shows that (20) indeed holds with $\mathbb{L}_\Psi$ replaced by $\mathbb{L}'_\Psi$, because $\mathbb{L}'_\Psi \subset \operatorname{im}(Q)$.

Lemma 18: Let $Q = Q^T \in \mathbb{R}^{n\times n}$ and let $\mathfrak{L}$ be a linear subspace of $\mathbb{R}^n$. Assume that $Q^\#$ is a symmetric generalized inverse of Q, i.e., $Q^\# = (Q^\#)^T$, $Q^\# Q Q^\# = Q^\#$, and $QQ^\# Q = Q$. Then $\mathfrak{L} \subset \operatorname{im}(Q)$ implies that $\operatorname{sign}(Q^\#|_{\mathfrak{L}}) = \operatorname{sign}(Q|_{Q^{-1}\mathfrak{L}})$.

This yields $\sigma_-(Q_\Psi|_{\mathcal{M}}) = \sigma_-(Q^\#|_{\mathbb{L}'_\Psi}) = \sigma_-(Q|_{Q^{-1}\mathbb{L}'_\Psi}) \leq \sigma_-(Q|_{Q^{-1}\mathbb{L}_\Delta^\perp}) = \sigma_-(Q|_{(Q\mathbb{L}_\Delta)^\perp})$, and hence we obtain the inequality

$$\sigma_-(Q_\Psi|_{\mathcal{M}}) \leq \sigma_-\left(Q|_{(Q\mathbb{L}_\Delta)^\perp}\right). \tag{21}$$

Now decompose, as in (6), the QDF $Q_\Psi|_{\mathcal{M}}$ as $Q_\Psi(v_3) = |F^+(d/dt)v_3|^2 - |F^-(d/dt)v_3|^2$ with $\operatorname{rowdim}(F^-) = \sigma_-(Q_\Psi|_{\mathcal{M}})$. Define, in analogy with (8), the controlled behavior in the singular case to be $\mathcal{K} = \mathcal{N} + \mathcal{F}^-$, with $\mathcal{F}^-$ the controllable part of the behavior $\{v_3 \in \mathcal{M} | F^-(d/dt)v_3 = 0\}$.

We now prove that $\mathcal{K} \in \mathfrak{L}^{\mathtt{v}}_{\text{cont}}$ satisfies i) $\mathcal{N} \subset \mathcal{K} \subset \mathcal{P}$, ii) $\mathcal{K}$ is Σ-dissipative on $\mathbb{R}_-$, and iii) $\mathtt{m}(\mathcal{K}) = \sigma_+(\Sigma)$. The proofs of i) and ii) are completely analogous to the regular case. The proof of iii) is more difficult. Indeed, the fact that $\mathcal{M}$ is in general a strict subset of $\mathcal{P} \cap \mathcal{N}^{\perp_\Sigma}$ makes it more unlikely that $\mathtt{m}(\mathcal{K})$ has a sufficiently high input cardinality, and a delicate estimate is required to establish this. Lemma 17 yields $\mathtt{m}(\mathcal{K}) \geq \mathtt{m}(\mathcal{N} + \mathcal{M}) - \sigma_-(Q_\Psi|_{\mathcal{M}})$. The last term has already been bounded in (21) by $\sigma_-(Q|_{(Q\mathbb{L}_\Delta)^\perp})$, which, in turn, was bounded in (19). We now obtain a suitable bound for $\mathtt{m}(\mathcal{N} + \mathcal{M})$ by obtaining a lower bound for $\mathtt{m}(\mathcal{M})$. We do this by invoking the driving variable (16) and output nulling representation (17) of $\mathcal{N}$ and $\mathcal{P}$. By using the relation between driving variable representations and output nulling representations of a behavior and its orthogonal complement, we obtain

$$\frac{d}{dt} z_{\mathcal{N}} = -A_{\mathcal{N}}^T z_{\mathcal{N}} + C_{\mathcal{N}}^T \Sigma v, \qquad 0 = B_{\mathcal{N}}^T z_{\mathcal{N}} - D_{\mathcal{N}}^T \Sigma v$$

as a driving variable representation of $\mathcal{N}^{\perp_\Sigma}$. With z defined by $z = (z_{\mathcal{N}}, x_{\mathcal{P}})$, this yields the output nulling representation $(d/dt)z = -F^T z + H^T \Sigma v$, $0 = G^T z - J^T \Sigma v$ of $\mathcal{P} \cap \mathcal{N}^{\perp_\Sigma}$. Note that $\mathcal{M}$ consists exactly of those $v \in \mathcal{P} \cap \mathcal{N}^{\perp_\Sigma}$ that yield $z(t) = (z_{\mathcal{N}}, x_{\mathcal{P}})(t) \in \mathcal{K}$ for all $t \in \mathbb{R}$. This yields the following output nulling representation of $\mathcal{M}$:

$$\frac{d}{dt} Kz' = -F^T Kz' + H^T \Sigma v, \quad 0 = G^T Kz' - J^T \Sigma v. \tag{22}$$

This representation allows to estimate $\mathtt{m}(\mathcal{M})$ using the following proposition.

Proposition 19: Let $(d/dt)x = Ax + Bw$, $0 = Cx + Dw$ be an output nulling representation of $\mathfrak{B} \in \mathfrak{L}^{\mathtt{w}}$. Then $\mathtt{m}(\mathfrak{B}) \geq \mathtt{w} - \mathrm{rank}([C \quad D])$.

The estimate of this lemma applied to (22) immediately yields

$$\mathtt{m}(\mathcal{M}) \geq q - \left(\mathrm{rank}\left(\begin{bmatrix} K & -F^TK & H^T\Sigma \\ 0 & G^TK & -J^T\Sigma \end{bmatrix}\right) - \mathrm{rank}(K)\right)$$

since the second term on the right-hand side equals the number of static relations in (22): the total number of relations involving $(v, z', (d/dt)z')$ minus those that contain $(d/dt)z'$. In terms of the matrix R introduced in (18) this yields $\mathtt{m}(\mathcal{M}) \geq q - (\mathrm{rank}(R) - \mathrm{rank}(K))$. Combining this with the identity $\mathtt{m}(\mathcal{N} + \mathcal{M}) = \mathtt{m}(\mathcal{N}) + \mathtt{m}(\mathcal{M}) - \mathtt{m}(\mathcal{N} \cap \mathcal{M})$, we obtain

$$\begin{aligned}
\mathtt{m}(\mathcal{K}) \geq\, & \mathtt{m}(\mathcal{N} + \mathcal{M}) - \sigma_-(Q_\Psi|_{\mathcal{M}}) \\
\geq\, & \mathtt{m}(\mathcal{N}) + q - (\mathrm{rank}(R) - \mathrm{rank}(K)) - \mathtt{m}(\mathcal{N} \cap \mathcal{M}) \\
& - \mathtt{m}(\mathcal{N}) + \mathtt{m}(\mathcal{N} \cap \mathcal{N}^{\perp_\Sigma}) - \sigma_-(\Sigma) \\
& + (\mathrm{rank}(R) - \mathrm{rank}(K)) \\
\geq\, & q - \sigma_-(\Sigma) + \mathtt{m}(\mathcal{N} \cap \mathcal{N}^{\perp_\Sigma}) - \mathtt{m}(\mathcal{N} \cap \mathcal{M}) \\
\geq\, & q - \sigma_-(\Sigma) = \sigma_+(\Sigma)
\end{aligned}$$

where we used that $\mathtt{m}(\mathcal{N} \cap \mathcal{N}^{\perp_\Sigma}) \geq \mathtt{m}(\mathcal{N} \cap \mathcal{M})$, since $\mathcal{M} \subset \mathcal{N}^{\perp_\Sigma}$. Thus, we get $\mathtt{m}(\mathcal{K}) \geq \sigma_+(\Sigma)$. Combining this with Proposition 2 we obtain $\mathtt{m}(\mathcal{K}) = \sigma_+(\Sigma)$

This completes the proof of Theorem 5 in the singular case.

C. Proof of Theorem 7

This theorem is an immediate consequence of the main result, and the inequalities $Q_{\Phi_{\mathcal{N}}}(v_1) \leq Q_{\Phi_{\mathcal{N}}^{\sup}}(v_1)$ for all $v_1 \in \mathcal{N}$, and $Q_{\Phi_{\mathcal{P}^\perp}^{\inf}}(v_2) \leq Q_{\Phi_{\mathcal{P}^\perp}}(v_2)$, for all $v_2 \in \mathcal{P}^\perp$.

D. Proof of Theorem 8

Let $X \in \mathbb{R}^{\mathtt{n}(\mathcal{K}) \times \mathtt{v}}[\xi]$ induce a minimal state map for $\mathcal{K}$. It can be shown that since $|\mathrm{col}(X_{\mathcal{N}}(d/dt)v_1, 0) + K^{\#}\mathrm{col}(Z_{\mathcal{N}}(d/dt)\Sigma v_3, X_{\mathcal{P}}(d/dt)v_3)|_K^2 \leq |v_1 + v_3|_\Sigma^2$ for $v_1 \in \mathcal{N}$, $v_3 \in \mathcal{F}^-$, there exists a matrix $L = L^T \in \mathbb{R}^{\mathtt{n}(\mathcal{K}) \times \mathtt{n}(\mathcal{K})}$ such that for $v_1 \in \mathcal{N}$, $v_3 \in \mathcal{F}^-$

$$\begin{aligned}
&\left|\mathrm{col}\left(X_{\mathcal{N}}\left(\frac{d}{dt}\right)v_1, 0\right)\right. \\
&\qquad \left. + K^{\#}\mathrm{col}\left(Z_{\mathcal{N}}\left(\frac{d}{dt}\right)\Sigma v_3, X_{\mathcal{P}}\left(\frac{d}{dt}\right)v_3\right)\right|_K^2 \\
&= \left|X\left(\frac{d}{dt}\right)(v_1 + v_3)\right|_L^2.
\end{aligned}$$

This implies $(d/dt)|X(d/dt)v|_L^2 \leq |v|_\Sigma^2$ for all $v \in \mathcal{K}$, so $|X(d/dt)v|_L^2$ is a storage function for $\mathcal{K}$. Using Σ-dissipativity of $\mathcal{K}$ on $\mathbb{R}_-$, $\mathtt{m}(\mathcal{K}) = \sigma_+(\Sigma)$, and proposition 12, yields $L > 0$. It follows that $\mathtt{n}(\mathcal{K}) = \mathrm{rank}(L)$. But $\mathrm{rank}(L)$ is less than or equal to the rank of the QDF $|\mathrm{col}(X_{\mathcal{N}}(d/dt)v_1, 0) + K^{-1}\mathrm{col}(Z_{\mathcal{N}}(d/dt)\Sigma v_3, X_{\mathcal{P}}(d/dt)v_3)|_K^2$ for $v_1 \in \mathcal{N}$, $v_3 \in \mathcal{F}^-$. This rank is bounded by $\mathtt{n}(\mathcal{N}) + \mathtt{n}(\mathcal{P})$, the dimension of K. This yields the bound on $\mathtt{n}(\mathcal{K})$.

VIII. Proofs of Auxiliary Results

In the interest of brevity, we often limit ourselves to giving the outline of the proofs.

Proof of Proposition 2: Factor $\Phi = F_+^TF_+ - F_-^TF_-$, with $(\mathrm{rowdim}(F_-), \mathrm{rowdim}(F_+)) = (\sigma_-(\Phi), \sigma_+(\Phi))$. Then $Q_\Phi(w) = |F_+w|^2 - |F_-w|^2$. Consider $\mathfrak{B}' = \{w \in \mathfrak{B} | F_+w = 0\}$. By lemma 17, $\mathtt{m}(\mathfrak{B}') \geq \mathtt{m}(\mathfrak{B}) - \sigma_+(\Phi)$. If $\mathtt{m}(\mathfrak{B}') > 0$, there exists $0 \neq w \in \mathfrak{B}' \cap \mathfrak{D}$. For this w, we also have $F_+w = 0$. Since $\mathrm{col}(F_-, F_+)$ is nonsingular, this yields $F_-w \neq 0$. Hence $Q_\Phi(w) = -|F_-w|^2 < 0$. This contradiction establishes the proposition.

Proof of Proposition 3: See [15, Prop. 5.4 and Th. 6.3].

Proof of Proposition 4: The (*if*) and (*only if*) part follow by considering the BDF $L_\Phi(w_1, w_2)$ as a QDF on $\mathfrak{B}_1 \times \mathfrak{B}_2$, and applying [15, Th. 3.1]. Finally, assume that $(d/dt)L_{\Psi_1}(w_1, w_2) = (d/dt)L_{\Psi_2}(w_1, w_2)$ for all $w_1 \in \mathfrak{B}_1$ and $w_2 \in \mathfrak{B}_2$. Since $\mathfrak{B}_1$ and $\mathfrak{B}_2$ are controllable, they can be represented using observable image representations, i.e., there exists left-invertible $M_1, M_2 \in \mathbb{R}^{\mathtt{w} \times \bullet}[\xi]$ such that $\mathfrak{B}_1 = \mathrm{im}(M_1(d/dt))$ and $\mathfrak{B}_2 = \mathrm{im}(M_2(d/dt))$. So, $(\zeta + \eta)M_1^T(\zeta)\Psi_1(\zeta, \eta)M_2(\eta) = (\zeta + \eta)M_1^T(\zeta)\Psi_2(\zeta, \eta)M_2(\eta)$. Therefore, $M_1^T(\zeta)\Psi_1(\zeta, \eta)M_2(\eta) = M_1^T(\zeta)\Psi_2(\zeta, \eta)M_2(\eta)$, which proves "essential" uniqueness.

Proof of Proposition 6: For the case $\mathfrak{B} = \mathfrak{C}^\infty(\mathbb{R}, \mathbb{R}^{\mathtt{w}})$, see [15, Th. 5.7]. The general case can be reduced to this one by an observable image representation of $\mathfrak{B}$.

Proof of Proposition 9: We leave the proof of this intuitive result to the reader.

Proof of Proposition 10: This result is proven in [9, Th. 6.2].

Proof of Corollary 11: The corollary states that the behavior $\mathfrak{B}_1 \times \mathfrak{B}_2$ is dissipative (lossless) with respect to the supply rate $w_1^T\Phi w_2$. Now apply Proposition 10.

Proof of Proposition 12: $\mathfrak{B} \in \mathfrak{L}_{\mathrm{cont}}^{\mathtt{w}}$ admits a full row rank kernel representation $R(d/dt)w = 0$ and an observable image representation $w = M(d/dt)\ell$. It follows that $w' = \Phi^{-1}R^T(-d/dt)\ell'$ is an observable image representation of $\mathfrak{B}^{\perp_\Phi}$. Let $\Phi'(\zeta, \eta) = M^T(\zeta)\Phi M(\eta)$ and $\Phi''(\zeta, \eta) = -R(-\zeta)\Phi^{-1}R^T(-\eta)$. Reference [15, Th. 10.2] yields $\int_{-\infty}^{\infty} Q_{\Phi'}(\ell)\, dt \geq 0$ for all $\ell \in \mathfrak{D}(\mathbb{R}, \mathbb{R}^{\mathtt{m}(\mathfrak{B})})$. Equivalently, $\int_{-\infty}^{\infty} Q_{\Phi''}(\ell')\, dt \geq 0$ for all $\ell' \in \mathfrak{D}(\mathbb{R}, \mathbb{R}^{\mathtt{m}(\mathfrak{B}^{\perp_\Phi})})$. Consequently, $\mathfrak{B}$ is Φ-dissipative if and only if $\mathfrak{B}^{\perp_\Phi}$ is $(-\Phi)$-dissipative. Part 1 follows.

By [15, Th. 6.4], $\mathfrak{B}$ is Φ-dissipative on $\mathbb{R}_-$ if and only if there exists $K = K^T > 0$ such that $|X(d/dt)w|_K^2$ is a storage function. By proposition 12, $-|Z(d/dt)\Phi w'|_{K^{-1}}^2$ is then a nonpositive storage function for $\mathfrak{B}^{\Phi_\perp}$ as a $(-\Phi)$-dissipative system. By proposition 3, this implies that $\mathfrak{B}^{\perp_\Phi}$ is $(-\Phi)$-dissipative on $\mathbb{R}_+$. This yields part 2.

Part 3 in the constant case immediately follows from Proposition 12. Parts 4 and 5 follow from [15, Th. 6.4 and 10.2 iv].

Proof of Lemma 13: Let L be a matrix of full column rank such that $\mathrm{im}(L) = \mathbb{L}$. Taking a suitable basis of $\mathbb{R}^n$, we may assume that

$$Q = \begin{bmatrix} Q_1 & 0 \\ 0 & 0 \end{bmatrix}, \qquad L = \begin{bmatrix} L_1 \\ N_1 \end{bmatrix}$$

with $Q_1 = Q_1^T$ nonsingular. Let L_2 be a matrix such that $L_2^T L_1 = 0$ and $\mathrm{rank}(L_1) + \mathrm{rank}(L_2) = \mathrm{rank}(Q)$. Then it is easily verified that

$$(Q\mathbb{L})^\perp = \mathrm{im}\left(\begin{bmatrix} Q_1^{-1}L_2 & 0 \\ 0 & I \end{bmatrix}\right).$$

According to [10, lemma 12.2], $\sigma_+(L_1^T Q_1 L_1) + \sigma_+(L_2^T Q_1^{-1} L_2) = \sigma_+(Q_1) - (\mathrm{rank}(L_1) - \mathrm{rank}(L_1^T Q_1 L_1))$ Using the fact that the rank of a symmetric matrix equals the sum of its negative and its positive eigenvalues, we obtain $\sigma_+(L_2^T Q_1^{-1} L_2) = \sigma_-(L_1^T Q_1 L_1) + \sigma_+(Q_1) - \mathrm{rank}(L_1)$. In this equality we obviously have $\sigma_+(Q_1) = \sigma_+(Q)$, $\sigma_-(L_1^T Q_1 L_1) = \sigma_-(Q|\mathbb{L})$, $\mathrm{rank}(L_1) = \dim(\mathbb{L}) - \dim(\ker(Q) \cap \mathbb{L})$, and finally

$$\begin{aligned}\sigma_+(L_2^T Q_1^{-1} L_2) &= \sigma\left(\begin{bmatrix} Q_1^{-1}L_2 & 0 \\ 0 & I \end{bmatrix}^T \begin{bmatrix} Q_1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} Q_1^{-1}L_2 & 0 \\ 0 & I \end{bmatrix}\right) \\ &= \sigma_+(Q|(Q\mathbb{L})^\perp).\end{aligned}$$

This proves the first part of the lemma. The second part is proven in a similar way.

Proof of Lemma 14: See, for instance, [7, section 3.6].

Proof of Lemma 15: We omit the straightforward proof.

Proof of Proposition 16: Let $w = M(d/dt)\ell$ be an observable image representation of $\mathfrak{B}$, and consider $\Phi'(\zeta, \eta) = M^T(\zeta)\Phi M(\eta)$. Note that $\mathtt{m}(\mathfrak{B}) = \mathrm{rowdim}(\Phi')$. Denote $\partial\Phi'(\xi) = \Phi'(-\xi, \xi)$. Clearly, $\partial\Phi'(d/dt)\ell = 0$, $w = M(d/dt)\ell$ is a representation of $\mathfrak{B} \cap \mathfrak{B}^{\perp_\Phi}$. Hence, $\mathtt{m}(\mathfrak{B} \cap \mathfrak{B}^{\perp_\Phi}) = \mathtt{m}(\mathfrak{B}) - \mathrm{rank}(\partial\Phi')$. By Φ-dissipativity of $\mathfrak{B}$ we have $\partial\Phi'(i\omega) \geq 0$ for all $\omega \in \mathbb{R}$, hence there exists $F \in \mathbb{R}^{\mathtt{m}(\mathfrak{B})\times\mathtt{m}(\mathfrak{B})}[\xi]$ such that $\partial\Phi'(\xi) = F^T(-\xi)F(\xi)$. Let $U(\xi)$ be any unimodular matrix such that $F(\xi) = [F_1(\xi) \quad 0]U(\xi)$, with F_1 full column rank, say $\mathtt{m}_1$. Note that $\mathtt{m}_1 = \mathrm{rank}(\partial\Phi)$, so $\mathtt{m}_1 = \mathtt{m}(\mathfrak{B}) - \mathtt{m}(\mathfrak{B} \cap \mathfrak{B}^{\perp_\Phi})$. We then have

$$\partial\Phi(\xi) = U^T(-\xi)\begin{bmatrix} F_1^T(-\xi)F(\xi) & 0 \\ 0 & 0 \end{bmatrix} U(\xi).$$

By [15, remark 5.13], $\Psi^{\inf}$ can be computed as follows. Factor $F_1^T(-\xi)F_1(\xi) = H_1^T(-\xi)H_1(\xi)$, with $H \in \mathbb{R}^{\mathtt{m}_1\times\mathtt{m}_1}[\xi]$ almost Hurwitz [i.e., nonsingular and with all the roots of $\det(H)$ in the closed left half of the complex plane]. Next, define

$$H(\xi) = \begin{bmatrix} H_1(\xi) & 0 \\ 0 & 0 \end{bmatrix} U(\xi).$$

Then $\partial\Phi'(\xi) = H^T(-\xi)H(\xi)$. Define $\Psi_1^{\inf}(\zeta, \eta) = (\Phi'(\zeta, \eta) - H^T(\zeta)H(\eta))/(\zeta + \eta)$. Let M^L be a left-inverse of M. Then $\Psi^{\inf} = M^L(\zeta)^T \Psi_1^{\inf}(\zeta, \eta) M^L(\eta)$ induces the smallest storage function for $\mathfrak{B}$ as an Φ-dissipative system. This implies that for all $w \in \mathfrak{B}$ we have

$$\begin{aligned}|w|_\Phi^2 - \frac{d}{dt}Q_{\Psi^{\inf}}(w) &= \left|H\left(\frac{d}{dt}\right)M^L\left(\frac{d}{dt}\right)w\right|^2 \\ &= \left|H_1\left(\frac{d}{dt}\right)U_1\left(\frac{d}{dt}\right)M^L\left(\frac{d}{dt}\right)w\right|^2\end{aligned}$$

where U_1 is the polynomial matrix consisting of the first $\mathtt{m}_1$ rows of U. Thus, the rank of this QDF on $\mathfrak{B}$ is equal to the rank of the polynomial matrix $H_1U_1M^L$. Since $H_1U_1M^L$ has full row rank, this rank is equal to $\mathrm{rowdim}(H_1U_1M^L) = \mathtt{m}_1 = \mathtt{m}(\mathfrak{B}) - \mathtt{m}(\mathfrak{B} \cap \mathfrak{B}^{\perp_\Phi})$. A similar argument applies to the largest storage function $\Psi^{\sup}$.

Finally, by Proposition 9, $\mathfrak{B} + \mathfrak{B}^{\perp_\Phi} = \mathfrak{C}^\infty(\mathbb{R}, \mathbb{R}^{\mathtt{w}})$ is equivalent to $\mathtt{m}(\mathfrak{B} \cap \mathfrak{B}^{\perp_\Phi}) = 0$.

Proof of Lemma 17: Let $R(d/dt)w = 0$ be a kernel representation of $\mathfrak{B}$. Then a kernel representation of $\mathfrak{B}'$ is given by $R(d/dt)w = 0$, $F(d/dt)w = 0$. This implies $\mathtt{p}(\mathfrak{B}') = \mathrm{rank}(\mathrm{col}(R, F)) \leq \mathrm{rank}(R) + \mathrm{rank}(F) \leq \mathtt{p}(\mathfrak{B}) + \mathrm{rowdim}(F)$. Therefore $\mathtt{m}(\mathfrak{B}') = \mathtt{w} - \mathtt{p}(\mathfrak{B}') \geq \mathtt{w} - \mathtt{p}(\mathfrak{B}) - \mathrm{rowdim}(F) = \mathtt{m}(\mathfrak{B}) - \mathrm{rowdim}(F)$.

Proof of Lemma 18: First note that, in general, if $Q = Q^T \in \mathbb{R}^{n\times n}$ and if $\mathcal{V} \subset \mathbb{R}^n$ is a linear subspace, then $\sigma_-(Q|_\mathcal{V}) = \sigma_-(V^TQV)$, the number of negative eigenvalues of V^TQV. Also note that the assumption $\mathfrak{L} \subset \mathrm{im}(Q)$ implies that $Q^{-1}\mathfrak{L} = Q^\#\mathfrak{L} + \ker(Q)$. Let L be any matrix such that $\mathrm{im}(L) = \mathfrak{L}$. Then we get $\sigma_-(Q|_{Q^{-1}\mathfrak{L}}) = \sigma_-(Q|_{Q^\#\mathfrak{L}+\ker(Q)}) = \sigma_-(Q|_{Q^\#\mathfrak{L}}) = \sigma_-((Q^\#L)^TQ(Q^\#L)) = \sigma_-(L^TQ^\#QQ^\#L) = \sigma_-(L^TQ^\#L) = \sigma_-(Q^\#|_\mathfrak{L})$. In the same way, we obtain $\sigma_+(Q|_{Q^{-1}\mathfrak{L}}) = \sigma_+(Q^\#|_\mathfrak{L})$.

Proof of Proposition 19: Assume without loss of generality that $[C \quad D]$ is of full-row rank. Denote this rank by $\mathtt{p}$. Consider the input-state-output system $(d/dt)x = Ax + Bw$, $y = Cx + Dw$. Elimination of x yields $P(d/dt)y = Q(d/dt)w$, with P square and $\det(P) \neq 0$, and $Q \in \mathbb{R}^{\mathtt{p}\times\mathtt{w}}[\xi]$. Then $\mathfrak{B}$ has kernel representation $Q(d/dt)w = 0$, whence $\dim(\mathfrak{B}) \geq \mathrm{coldim}(Q) - \mathrm{rowdim}(Q) = \mathtt{w} - \mathtt{p} = \mathtt{w} - \mathrm{rank}([C\ D])$.

IX. REMARKS

In this section, we deal with some of the assumptions that have been made on the plant, the controlled behaviors, the controllers, and the signal spaces.

1) Controllability: Throughout we assumed, where convenient, that the systems are controllable. The main reason for this is that—at this point—we have no satisfactory definition of dissipativeness for uncontrollable systems. However, for the synthesis problem, there are *ad hoc* ways of dealing with lack of controllability. We explain this in the case that $Q_\Sigma(v) = |d|^2 - |f|^2$.

Assume $\mathcal{N}, \mathcal{P} \in \mathfrak{L}^{\mathtt{v}}$, $\mathcal{N} \subset \mathcal{P}$, given. Consider finding a $\mathcal{K} \in \mathfrak{L}^{\mathtt{v}}$ such that: i) $\mathcal{N} \subset \mathcal{K} \subset \mathcal{P}$; ii) $\mathtt{m}(\mathcal{K}) = \dim(d)$; iii) the controllable part of $\mathcal{K}$ is Σ-dissipative on $\mathbb{R}_-$, and iv) $\mathcal{K}$ is stabilizable (meaning that for all $v \in \mathcal{K}$ there exists $v' \in \mathcal{K}$ with $v'(t) \to 0$ for $t \to \infty$, such that $v \wedge v' \in \mathcal{K}$). These conditions are equivalent to i)′ implementability of $\mathcal{K}$ (no controllability assumptions are made in theorem 1), ii)′ in $\mathcal{K}$, d is input, f is output, and the transfer function from d to f in $\mathcal{K}$, $G_{d\mapsto f}$, satisfies $||G_{d\mapsto f}||_{\mathcal{H}_\infty} \leq 1$, and iii)′ $(d, f) \in \mathcal{K}$, and $d|_{\mathbb{R}_+} = 0$ implies $f(t) \to 0$ for $t \to \infty$. In other words, the controlled system is (externally) stable.

The above problem is solvable if and only if i) the conditions of theorem 5 hold with $\mathcal{N}$ and $\mathcal{P}$ replaced by their controllable part, and ii) $(d, f) \in \mathcal{N}$ and $d|_{\mathbb{R}_+} = 0$ implies $f(t) \to 0$ for $t \to \infty$. $\mathcal{K}$ can then be obtained by first constructing a $\mathcal{K}_0$ based on the controllable parts of $\mathcal{N}$, $\mathcal{P}$, and taking $\mathcal{K} = \mathcal{K}_0 + \mathcal{N}$. Note that a stability condition on $\mathcal{N}$ enters, but stabilizability of $\mathcal{P}$ does not. However, when considering implementability by regular or feedback (see [14]) controllers, a more careful choice of the autonomous part of $\mathcal{K}$ needs to be made, and stabilizability of $\mathcal{P}$ does enter.

2) Smoothness: Throughout, we assume that the system trajectories are in $\mathfrak{C}^\infty$, even, when dealing with dissipativeness, in $\mathfrak{B} \cap \mathfrak{D}$. Assuming $\mathfrak{C}^\infty$ solutions of the differential equations, instead of, say, $\mathfrak{L}_2^{\mathrm{loc}}$ solutions, avoids mathematical technicalities, as dealing with distributions (also in the QDF′), that are not germane to our purposes. Requiring in the definition of dissipativeness $\int_{-\infty}^{\infty} Q_\Phi(w)\,dt \geq 0$ for $w \in \mathfrak{B} \cap \mathfrak{D}$, instead of a broader class of solutions [with $Q_\Phi(w)$ properly interpreted] means that systems are—but only in principle—more likely dissipative. Hence the conditions of theorem 5 are more likely satisfied. As explained in the context of disturbance attenuation, we nevertheless obtain that the transfer function in $\mathcal{K}$ has $\mathcal{H}_\infty$-norm ≤ 1. As is well-known, this frequency domain implies contractiveness of $\mathcal{K}$ for $d \in \mathcal{L}_2$, for periodic d, etc.

3) Algorithms: The results of this paper open up many algorithmic questions: how to verify dissipativeness, how to construct storage functions, how to obtain a convenient representation of $\mathcal{K}$ or $\mathcal{C}$ starting from a (for example, general latent variable) representation of $\mathcal{P}_{\mathrm{full}}$). In an earlier draft, algorithms were discussed, but they have been deleted because of length limitations. We will deal with algorithms in a sequel paper.

4) Nonlinear Systems: Generalizing the results on implementability, and on the synthesis of dissipative systems to nonlinear systems appears to be a grand challenge, but beyond the scope of the present paper. The question whether there exists a $\mathcal{K} \in \mathfrak{L}^{\mathtt{v}}$ that solves our synthesis problem (with $\mathcal{N}, \mathcal{P} \in \mathfrak{L}^{\mathtt{v}}$) whenever there exists any (nonlinear, time-varying, etc.) such $\mathcal{K}$, appears more accessible. It is easy to see that $\mathcal{N}$ must still be dissipative, but how to conclude conditions on $\mathcal{P}^{\perp_\Sigma}$ from $\mathcal{K} \subset \mathcal{P}$, is more elusive.

X. Conclusion

We have presented a complete solution, in the context of controllable linear differential systems of making a plant dissipative by attaching a controller to it. In our approach, the plant is specified in a representation-free manner by its behavior. The existence of a controller that meets the specifications depends on dissipativity properties of the hidden behavior and of the uncontrolled plant behavior, combined with a subtle coupling condition involving storage functions.

We viewed control as interconnection. This formulation broadens the scope of control theory from a practical point of view. In this setting, control becomes the design of subsystems, aimed at enhancing the performance of the over-all system. Feedback emerges as a special case, and, we hasten to emphasize, a very important one.

We have discussed implementation of a controlled behavior by a controller that acts on the control variables only. We have shown that a behavior is implementable if and only if it is contained in the plant behavior and contains the hidden behavior. The plant behavior consists of the trajectories of the to-be-controlled variables that are possible before control is applied. The hidden behavior consists of the trajectories of the to-be-controlled variables that remain possible when the control variable are equal to zero, i.e., those trajectories compatible with vanishing control variables.

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